Legendre Polynomials

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Lectures in Mathematics

The following lecture introduces the Legendre polynomials. It includes their derivation, and the topics of orthogonality, normalization, and recursion.

I. General Formula

We start with a solution to the Laplace equation in 3 - dimensional space:

$$\Delta U = 0 \qquad (1)$$

A solution is:

$$U = \frac{1}{|\vec{r} - \vec{r_0}|} \qquad |\vec{r_0}| = 0 \qquad (2)$$

Let us introduce the spherical coordinate system. The z-axis is along $\vec{r_0}$.
In this system the axial symmetric Laplace equation looks as follows:

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \sin(\theta) \frac{\partial U}{\partial \theta} \qquad (3)$$

Solution (2) is:

$$U = \frac{1}{\sqrt{1 - rx + z^2}} \qquad (4)$$

$x = \cos\theta$

Equation (3) can be rewritten as follows:

$$\frac{1}{r^2}\ \frac{\partial}{\partial r} r^2 \frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial x}(1\text{-}x^2)\frac{\partial U}{\partial x} = 0 \qquad (5)$$

For (r<1) the function U(r,x) can be presented by a power series:

$$U = \sum_{k=0}^{\infty} r^n P_n(x) \qquad (6)$$

***P_n* (*x*)** are polynomials. They are called "**Legendre's Polynomials"**.

Let x=1 in Equation (4), Then,

$$U = \frac{1}{\sqrt{\left(1 - r^2\right)^2}} = \frac{1}{1-r} = 1 + r + r^2 + \ldots \qquad (7)$$

Comparing (7) and (6) we get

$P_n(1) = 1$

By pluggin (6) into (5) we find that $P_n(x)$ satifies the equation :

$$\frac{\partial}{\partial x}\left(1 - x^2\right)\frac{\partial P_n}{\partial x} - n(n+1)\,P_n = 0 \qquad (8)$$

This is **Legendre's Equation**.

We can seek its solution in the form of a power series:

$$P_n = \sum a_k\, x^k \qquad (9)$$

By plugging (9) into (8) we get:

$$(k+2)(k+1)a_{k+2} + [n(n+1) - k(k+1)]a_k = 0$$

$$a_{k+2} = -\frac{(n-k)(n+k+1)}{(k+2)(k+1)} a_k \tag{10}$$

Condition (10) is satisfied if P_n(x) is presented by the series

$$P_n(\mathrm{x}) = \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k (2n-2k)!}{k!\,(n-k)!\,(n-2k)!} x^{n-2k} \tag{11}$$

The series (11) is terminated as far as one of the factorials in the denominator becomes infinite

Remember that

$$\mathrm{n}! = \int_0^{\infty} t^{n-1} e^{-t}\, dt$$

Hence, n! = ∞, if n is a negative integer

Relation (10), proving satisfaction of equation (8) can be checked by the use of (11) immediately.

To be sure that P(1) = 1, we formulate the following:

Theorem

P_n(x) can be presented by the Rodriges formula:

$$P_n = \frac{1}{2^n n!} \frac{\partial^n}{\partial x^n} \left(x^2 - 1\right)^n \tag{12}$$

Indeed $\left(x^2 - 1\right)^n = \sum_{k=0}^{\infty} \frac{n!}{k!\,(n-k)!} (-1)^k x^{2(n-k)}$

$$\frac{\partial^n}{\partial x^n} x^{2(n-k)} = \frac{(2n-2k)!}{(n-2k)!} x^{2(n-k)}$$

By comparison with (11) we obtain the desired result.

Then,

$$P_n(\mathrm{x}) = \frac{1}{2^n n!} \frac{\partial^2}{\partial x^n} (x-1)^n (x+1)^n \tag{13}$$

Let $x \to 1$, In this limit we should differentiate only $(x-1)^n$, otherwise we get zero.

$$P_n(1) = \frac{1}{2^n n!} \mathrm{n}!\,(1+1)^n = 1$$

II. Orthogonality

Let us consider equations for P_n, P_m

(14)

$$\begin{cases} \frac{\partial}{\partial x}\left(x^2-1\right)\frac{\partial P_n}{\partial x}+n(n+1)\,P_n=0 & P_n \\ \frac{\partial}{\partial x}\left(x^2-1\right)\frac{\partial P_m}{\partial x}+m(m+1)\,P_m=0 & P_m \end{cases} \qquad (14)$$

From (14) we get

$$P_m\frac{\partial}{\partial x}\left(x^2-1\right)\frac{\partial P_n}{\partial x}-P_n\frac{\partial}{\partial x}\left(x^2-1\right)\frac{\partial P_m}{\partial x}+[n(n+1)-m(m+1)]P_nP_m$$

1. Orthogonality

Let $P_n(x)$, $P_m|y)$ - two polynomials, satisfy the following equations:

$$\frac{\partial}{dx}\left(x^2-1\right)\frac{\partial P_n}{\partial x}+n(n+1)\,P_n=0 \qquad (1)$$

$$\frac{\partial}{\partial x}\left(x^2-1\right)\frac{\partial P_m}{\partial x}+m(m+1)\,P_m=0 \qquad (2)$$

Let us multiply (1) by P_m and (2) by P_n and subtract, we get:

$$P_m\frac{\partial}{\partial x}\left(x^2-1\right)\frac{\partial P_n}{\partial x}-P_n\frac{\partial}{\partial x}\left(x^2-1\right)\frac{\partial P_m}{\partial x}+[n(n+1)-m(m+1)]P_nP_m=0 \qquad (3)$$

Equation (3) can be presented as follows:

$$\frac{\partial}{dx}\left(x^2-1\right)\left(P_m\frac{\partial P_n}{\partial x}-P_n\frac{\partial P_m}{\partial x}\right)+[n(n+1)-m(m+1)]P_nP_m=0 \quad (4)$$

After integration of (4) by x we get:

$$\left(1-x^2\right)\left(P_m\frac{\partial P_n}{\partial x}-P_n\frac{\partial P_m}{\partial x}\right)\Big|_{-1}^{+1}+[n(n+1)-m(m+1)]\int_{-1}^{1}P_n\,P_m\,dx=0$$

Hence,

$$\boxed{\int_{-1}^{1}P_n\,P_m\,dx=0}$$

2. Normalization

U(x,r) = $\frac{1}{\sqrt{1-2\,\text{xr}+r^2}} = \sum_{n=0}^{\infty} r^n P_n(\text{x})$

$U^2(\text{x,r}) = \frac{1}{1-2\,\text{xr}+r^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r^{n+m} P_n(\text{x}) P_m(\text{x})$

After integration by x we get:

$\int_{-1}^{1} U^2(\text{x,r})\text{dx} = \sum_{m=0}^{\infty} r^{2n} \int_{-1}^{1} P_n{}^2(\text{x})\text{dx}$

$\int_{-1}^{1} U^2\text{dx} = \int_{-1}^{1} \frac{dx}{1-2\,\text{xr}+r^2} = -\frac{1}{2r}\int_{-1}^{1} \frac{dx}{x-\frac{1}{2}\left(r+\frac{1}{2}\right)} = -\frac{1}{2r}\ln(x-\frac{1}{2}\left(r+\frac{1}{2}\right))\Big|_{-1}^{+1} = -\frac{1}{2}\ln\left(\frac{1-r}{1+r}\right)^2$

$=\frac{1}{r}(\ln(1+\text{r}) - \ln(1-\text{r})) = \frac{2}{r}(\text{r} + \frac{r^3}{3} + \frac{r^5}{5} + \ldots) = 2\sum_{n=0}^{\infty} \frac{r^{2n}}{2n+1}$

Hence,

$$\boxed{\int_{-1}^{1} P_n{}^2(x)\, dx = \frac{2}{2n+1}}$$

3. Recursion Relations

$\text{U} = \frac{1}{\left(1-2\,\text{xr}+r^2\right)^{\frac{1}{2}}}$

$\frac{\partial \text{U}}{\text{dr}} = \frac{\text{x}-\text{r}}{1-2\,\text{xr}+\text{r}^2}\text{U}$

$\left(1-2\,\text{xr}+\text{r}^2\right)\frac{\partial \text{U}}{\text{dr}} = (\text{x-r})\text{U}$

$\text{U} = \sum_{r=0}^{\infty} r^n P_n(\text{x})$

$\frac{\partial \text{U}}{\text{dr}} = \sum_{n=0}^{\infty} \text{nr}^{n-1} P_n(\text{x}) = \sum_{n=0}^{\infty} (n+1)\, r^n P_{n+1}(\text{x})$

$\text{r}\frac{\partial \text{U}}{\text{dr}} = \sum \text{nr}^n P_n(\text{x})$

$r^2 \frac{\partial \text{U}}{\text{dr}} = \sum_{n=0}^{\infty} \text{nr}^{n+1} P_n(\text{x}) = \sum_{n=1}^{\infty} (n-1)\, r^n P_{n-1}(\text{x})$

$\text{rU} = \sum_{n=0}^{\infty} r^{n+1} P_n(\text{x}) = \sum_{n=1}^{\infty} r^n P_{n-1}(\text{x})$

Collecting all together we get:

$\sum_{n=1}^{\infty} (n+1) P_{n+1} - \text{x}((2\text{n}+1)P_n + \text{n}P_{n-1}(x))r^n = 0$

Finally, we get the recursion relation

$$\boxed{(n+1)\, P_{n+1} - x\,(2\,n+1)\, P_n + n\, P_{n-1}(x) = 0}$$

Then,

$$\frac{\partial U}{dx} = \frac{r}{1-2\,xr+r^2}\,U$$

$$\frac{\partial U}{dr} = \frac{x-r}{1-2\,xr+r^2}\,U$$

$$(x-r)\frac{\partial U}{dx} = r\frac{\partial U}{dr}$$

$$r\frac{\partial U}{dr} = \sum n r^n P_n(x)$$

$$\frac{\partial U}{dx} = \sum r^n P_n'(x)$$

$$x\frac{\partial U}{dx} = \sum r^n x P_n'(x)$$

$$-r\frac{\partial U}{dr} = -\sum r^{n+1} P_n'(x) = -\sum r^n P_{n-1}'(x)$$

Thus, we get

$$\boxed{P_n' - xP_{n-1}' = nP_{n-1}}$$