A&M lectures, 2014

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Introduction to representation theory

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This is a series of lectures for the A&M particle theory students, prepared in the spring of 2014.

Sources

Throughout the course we will mainly follow

1. (phys) R. Cahn's book "Semi-Simple Lie Algebras and Their Representations."
2. (math) W. Fulton & J. Harris, "Representation Theory. A First Course."

There are also many other excellent references. We recommend

1. books by Humphreys, Jacobson, and Helgason for a complete mathematics exposition
2. on-line lecture notes by Robert Bryant on Lie Groups and symplectic geometry : `www.math.duke.edu/~bryant/ParkCityLectures.pdf`
3. Slansky, Phys. Rep. 79 (1981) 1–128 for an invaluable set of tables
4. LieArt, a Mathematica package by Feger and Kephart, ArXiv:1206.6379, which gives nice tables and introduction to their package
5. LiE, a great stand-alone package by Cohen, van Leewen, and Lisser
6. McKay & Patera : hard to get but very complete tables, combined with a very concise presentation of useful results and concepts.

Format

- We meet for lecture at 4pm on Fridays in 108 MIST.
- HW exercises are provided, and their completion is encouraged but will not be enforced.

1 Introduction to basic structures

motivation from physics

Symmetries play an essential role in theoretical physics, whether by constraining possible interactions, predicting new particles, or as fundamental structures that organize the natural world. When meeting a symmetry, we classify it in two ways:

- gauge vs global. A gauge symmetry is a redundancy of description, typically introduced to make some other symmetry manifest; e.g., the vector potential of E&M, with its gauge transformations, makes Lorentz invariance manifest). A global symmetry, on the other hand, relates distinct physical states with identical properties; e.g., SO(3) rotations of experimental apparatus in a rotationally-invariant background relate identical states.

- discrete vs continuous. If a symmetry group G has finite order $|G| < \infty$, it is discrete, while a continuous group has infinite order as a set. In the latter case, it is natural in the physics context that the set has a differential structure as well — it is not only continuous, but also differentiable.

Lie groups describe continuous symmetries relevant for theoretical physics, and with that we turn to our first definition:

Definition 1.1. A Lie group (LG for short) is a group G that is also a manifold. In particular, G is a C^∞ manifold, where the standard group structures of associative product $\times : G \times G \to G$ and inverse $\imath : G \to G$ are C^∞ maps of manifolds.

Instead of the order, we now have another useful way of characterizing the "size" of a LG, i.e. its dimension as a manifold. We can then usefully distinguish LGs with $\dim G = n$ and $\dim G = \infty$. The latter play an important role in physics. For instance, the group of gauge symmetries of Maxwell or Yang-Mills theories is infinite-dimensional, and certain theories can even exhibit infinite-dimensional global symmetries, such as the Virasoro symmetry of two-dimensional conformal field theory. However, in these lectures we will stick to finite-dimensional LGs.

Real manifolds have local structure of open sets $U_\alpha \simeq \mathbb{R}^n$, and on overlaps $U_\alpha \cap U_\beta \neq \emptyset$ they are glued by C^∞ isomorphisms $f_{\alpha\beta} : U_\alpha \to U_\beta$. *Complex manifolds* are a bit more special. They can be given local structure $U_\alpha \simeq \mathbb{C}^n$, and the patching data on $U_\alpha \cap U_\beta \neq \emptyset$ is given in terms of holomorphic isomorphisms $f_{\alpha\beta} : U_\alpha \to U_\beta$. We can thus naturally consider

Definition 1.2. A complex LG is a group that is a complex manifold, with product and inverse given by holomorphic maps.

While every complex LG is a real LG, the converse is of course not true. It is important to note that the product structure being holomorphic is crucial. For instance, by a classic result of Wang [1], every even-dimensional LG is a complex manifold, but the product is not holomorphic with respect to that complex structure. For instance, $\mathrm{SU}(2) \times \mathrm{SU}(2) = S^3 \times S^3$ is a complex manifold, but it is not a complex LG.

It is nice to have a notion of when two LGs are the same, and towards that goal we have

Definition 1.3. A LG homomorphism is a differentiable map $\varphi : G \to H$ between two LGs that respects the product structures, i.e. $\varphi(g \times_G g') = \varphi(g) \times_H \varphi(g')$, and $\varphi(\imath_G(g)) = \imath_H(\varphi(g))$ for all $g, g' \in G$. The map is a LG isomorphism if there exists an inverse LG homomorphism $\varphi^{-1} : H \to G$.

a glimpse of global structure

When considered as manifolds, LGs have a rich topology and geometry. They provide many explicit examples of interesting geometries, and that was one of the main reasons for their explorations by mathematicians in the late 19th and 20th centuries. Among the great achievements are classification results of various sorts, as well as many known homotopy/homology groups of various LGs. There are many famous names associated to this story: Bott, Borel, Cartan, Dynkin, Killing, Lie, Serre, just to name a few that come to mind. In this course we will only lightly touch upon the interesting global structure. Instead, we will concentrate on the local geometry of a LG. This is a very useful starting point for the following reason: there is a direct correspondence between the local geometry of LG with its Lie Algebra (LA) structure, and this correspondence determines the LG up to global geometric properties.

Just so that these "global" properties do not sound too mystical, let's give what's probably already a familiar example. The LGs SU(2) and SO(3) have isomorphic Lie Algebras, but globally they are distinct. While $\mathrm{SU}(2) = S^3$ is a simply connected manifold (i.e. $\pi_1(\mathrm{SU}(2)) = 0$), $\mathrm{SO}(3) = \mathbb{RP}^3$ is not simply connected: $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$.[1] In fact, there is a $2:1$ LG homomorphism $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$, which identifies antipodal points on S^3 to produce $\mathbb{RP}^3$. This is sometimes abbreviated by saying $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$.

The reader has undoubtedly encountered the notion of a subgroup. Similarly, we have

Definition 1.4. A Lie subgroup of a LG G is a subgroup $H \subset G$ that is also a submanifold.

Let's take a little break from definitions to give the prototypical example of a LG.

Example 1.5. Let V be a vector space (for us $V \simeq \mathbb{R}^n$ or $V \simeq \mathbb{C}^n$, but more general base fields are possible). Then $G = \mathrm{GL}(V)$ is the group of general linear transformations on V. For our two examples, G will be isomorphic as a LG to either $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$— $n \times n$ matrices valued in $\mathbb{R}$, $\mathbb{C}$, respectively, with $\det \neq 0$.

$\mathrm{GL}(V)$ is fundamental in the theory of LGs for the following reason.

Definition 1.6. A representation of G is a LG homomorphism $\rho : G \to \mathrm{GL}(V)$. V is then known as the "representation space."

Definition 1.7. A representation ρ (often summarized as "rep") is said to be faithful if ρ is injective. It is almost faithful if ρ has a 0-dimensional kernel.

$$\ker \rho = \{ g \in G \mid \rho(g) = \mathbb{1} \} .$$

A faithful representation has $\ker \rho = e$, the identity element of G, while for an almost faithful representation $\ker \rho$ is a discrete subgroup of G.

We now state the reason for why $\mathrm{GL}(V)$ is so basic to LG theory:

Theorem 1.8 (Ado/Iwasawa)**.** *Every LG has an almost faithful representation.*

[1] Recall that π_1, alternatively known as the fundamental group or the first homotopy group, measures the contractibility of loops in a space. Loosely speaking, for a connected manifold X, $\pi_1(X)$ is the group of maps $S^1 \to X$ up to homotopy, i.e. smooth deformations. So, for instance, $\pi_1(S^1) = \mathbb{Z}$, with the integers counting the number of times the circle is wrapped. A discrete π_1, such as $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$, signifies a non-trivial loop that cannot be contracted by itself, while the loop obtained by doubling the original loop can be contracted to a trivial map. A thorough presentation may be found in [2]; an accessible physics presentation is given by Weinberg in volume 2.

Basic examples: matrix Lie groups

We will now list some basic examples of LGs that will be our companions in most of the course. Arguably the most "hands-on" set of LGs is obtained as matrix Lie groups.

Definition 1.9. A matrix LG is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Example 1.10. $\mathrm{GL}(n, \mathbb{C})$ is a matrix LG.

$$GL(n, \mathbb{C}) \longrightarrow \mathrm{GL}(2n, \mathbb{R})$$

$$A + iB \longrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

Exercise 1.11. Show that the above is a LG homomorphism.

Here is an example of the sort of LG we will typically avoid:

Example 1.12.

$$B_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}) \right\} \ ,$$

is the LG of 2×2 upper-triangular invertible matrices.

A more familiar class of examples is provided by the special linear groups:

Example 1.13 (special linear groups)**.**

$$\mathrm{SL}(n, \mathbb{R}) = \{ M \in \mathrm{GL}(n, \mathbb{R}) \ \ | \det M = 1 \} \ ,$$
$$\mathrm{SL}(n, \mathbb{C}) = \{ M \in \mathrm{GL}(n, \mathbb{C}) \ \ | \det M = 1 \} \ .$$

Normal forms of complex matrices

In this little section we remind the reader of some classic results on the theory of complex matrices. We state results without proofs, which may be found in [3].

Lemma 1.14. *Let S be a unitary anti-symmetric $2n \times 2n$ matrix. Then $S = UJ\,{}^tU$, where U is unitary and*

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \ .$$

Lemma 1.15. *Let S be a unitary symmetric matrix. Then $S = U\,{}^tU$, where U is unitary.*

Lemma 1.16. *Let M be a complex anti-symmetric matrix. Then $M = UX\,{}^tU$, where U is unitary and*

$$X = diag(\mu_1, \mu_2, \ldots, \mu_n) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \mu_i \in \mathbb{R}_{\geq 0} \ .$$

Lemma 1.17. *Let M be a complex symmetric matrix. Then $M = UX\,{}^tU$, where X is diagonal, real, and non-negative.*

Lemma 1.18. *Let M be a complex matrix. Then $M = UXV$, where U, V are unitary and X is diagonal, real, and non-negative.*

Matrix groups preserving a non-degenerate form Q.

Fix V over a base field $k = \mathbb{R}$ or $k = \mathbb{C}$, and a non-degenerate pairing $V \times V \to k$. We can then ask for subgroups of $\mathrm{GL}(V)$ that preserve this pairing. This turns out to to all of the favorite examples of "classical" LGs.

Example 1.19 (real orthogonal groups)**.** Let Q by a symmetric bilinear form on $\mathbb{R}^n$ of signature p, q. Then by a change of basis we can without loss of generality take

$$Q = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \ .$$

The orthogonal group of signature (p, q) is defined as

$$\mathrm{O}(p,q) = \{M \in \mathrm{GL}(n,\mathbb{R}) \ \mid {}^tMQM = Q\} \ .$$

The special orthogonal group is then $\mathrm{SO}(p,q) = \mathrm{O}(p,q) \cap \mathrm{SL}(p+q,\mathbb{R})$. If the signature is $n, 0$, we will simply write these groups as $\mathrm{O}(n)$ and $\mathrm{SO}(n)$.

Example 1.20 (complex orthogonal groups)**.** Let Q be a symmetric bilinear non-degenerate form on $\mathbb{C}^n$. From our matrix theorems it is then clear that up to $\mathrm{GL}(n,\mathbb{C})$ transformations we can take $Q = \mathbb{1}_n$ (i.e. there is no notion of signature). We then define the complex orthogonal group

$$\mathrm{O}(n,\mathbb{C}) = \{M \in \mathrm{GL}(n,\mathbb{C}) \ \mid {}^tMQM = Q\} \ ,$$

as well as the special orthogonal group is then $\mathrm{SO}(n,\mathbb{C}) = \mathrm{O}(n,\mathbb{C}) \cap \mathrm{SL}(n,\mathbb{C})$.

Next we introduce a group familiar from Hamiltonian mechanics.

Example 1.21. Let Q be an antisymmetric[2] form on $\mathbb{R}^{2n}$. Then up to $\mathrm{GL}(n,\mathbb{R})$ we have

$$Q = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \ ,$$

i.e. Q is a symplectic structure on $\mathbb{R}^{2n}$, and we define

$$\mathrm{Sp}(2n,\mathbb{R}) = \{M \in \mathrm{GL}(2n,\mathbb{R}) \ \mid {}^tMQM = Q\} \ .$$

In an analogous way we can construct $\mathrm{Sp}(2n,\mathbb{C})$.

Exercise 1.22. Show that $\det M = 1$ for all $M \in \mathrm{Sp}(2n,\mathbb{R})$.

[2]Mathematicians often say "skew-symmetric," and we might occasionally ape them.

We now come to what is perhaps the most familiar and friendly LG.

Example 1.23. Let Q be a Hermitian positive-definite form on $\mathbb{C}^n$. Up to $\mathrm{GL}(n,\mathbb{C})$ we have $Q = \mathbb{1}_n$, and we define the unitary group

$$\mathrm{U}(n) = \{M \in \mathrm{GL}(n,\mathbb{C}) \ \mid M^\dagger Q M = Q\} \ .$$

Its special unitary version is $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C})$.

Exercise 1.24. Show that $\mathrm{SU}(2n)$ is not a complex LG.

Exercise 1.25. Show that $\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n,\mathbb{R})$, i.e. $\mathrm{U}(n)$ is the group of transformations that preserves both a symmetric form and a compatible symplectic form.

Our final entry in the matrix LG catalog for the day is the unitary symplectic group:

Example 1.26. Let Q be Hermitian positive definite and J anti-symmetric and non-degenerate, with $[Q,J] = 0$ on $\mathbb{C}^{2n}$. That is, without loss of generality $Q = \mathbb{1}_{2n}$, and J is a symplectic structure as above. Then $\mathrm{Sp}(n) = USp(2n) = \mathrm{U}(2n) \cap \mathrm{Sp}(2n,\mathbb{C})$.

A little compare and contrast

Having built up a little catalog of LGs, let us compare and contrast some of their properties.

G	connected?	π_1	center	compact?
$\mathrm{O}(n,\mathbb{R})$	no	$\mathbb{Z}_2$	$\mathbb{Z}_2$	yes
$\mathrm{SO}(n,\mathbb{R})$	yes	$\mathbb{Z}_2$ ($\mathbb{Z}$ for n=2)	1, n odd; $\mathbb{Z}_2$, n even	yes
$\mathrm{SL}(n,\mathbb{R})$	yes	$\mathbb{Z}_2$ ($\mathbb{Z}$ for n=2)	1, n odd; $\mathbb{Z}_2$, n even	no
$\mathrm{SL}(n,\mathbb{C})$	yes	trivial	$\mathbb{Z}_n$	no
$\mathrm{SU}(n)$	yes	trivial	$\mathbb{Z}_n$	yes

Exercise 1.27. Determine the dimensions of all the matrix LGs we described.

2 From LG to LA and back

In this lecture we will identify the structure that fixes the local geometry of a LG as the Lie algebra (abbreviated LA). We will see how every LG gives rise to a LA and every LA gives rise to a LG. This will give us the framework for splitting questions about LGs into local issues and global issues, and most of the following lectures will be devoted to exploring the local structure.

We begin with the abstract definition of a LA.

Definition 2.1. A Lie algebra is a vector space $\mathfrak{g}$ over a field k (for us $k = \mathbb{R}$ or $k = \mathbb{C}$) with a bilinear map, known as the Lie bracket,

$$[\bullet, \bullet] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \ ,$$

satisfying

1. skew symmetry: $[x, y] + [y, x] = 0$;
2. Jacobi identity: $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$

for all $x, y, z \in \mathfrak{g}$.

Before we get into more details, let us give two examples of such a structure. These are probably very familiar to the reader.

Example 2.2. The algebra $\mathfrak{gl}(n, \mathbb{C})$ has $n \times n$ complex matrices as the vector space, i.e. $\dim_{\mathbb{C}} \mathfrak{gl}(n, \mathbb{C}) = n^2$, and the bracket is simply the matrix product: $[x, y] = xy - yx$. Note that any matrix X in a small neighborhood of $\mathbb{1} \in \mathrm{GL}(n, \mathbb{C})$ can be written as $X = \mathbb{1} + x$ for some $x \in \mathfrak{gl}(n, \mathbb{C})$. More generally, we can consider $\mathfrak{gl}(V)$ for any real or complex vector space V.

Example 2.3. The algebra $\mathfrak{u}(n)$ arises by linearizing unitary matrices. Recall that $X \in \mathrm{U}(n)$ means $X^\dagger X = \mathbb{1}$, and linearizing $X = \mathbb{1} + x$ implies $x + x^\dagger = 0$, i.e. x is an $n \times n$ anti-Hermitian matrix.[3] So, we define the algebra $\mathfrak{u}(n)$ as the set of $n \times n$ anti-Hermitian matrices, with the bracket again defined by the matrix product: $[x, y] = xy - yx$.

The Lie bracket of the last example satisfies a non-trivial consistency condition: $[x, y]$ is anti-Hermitian even though in general $(xy)^\dagger \neq -xy$, i.e. the matrix product is not a well-defined operation on the LA.

We have an obvious definition that allow us to map LAs to each other and to check whether two LAs are isomorphic: A LA homomorphism is a linear map of vector spaces $\varphi : \mathfrak{g} \to \mathfrak{h}$ that preserves the Lie bracket. Isomorphic LAs are related by an invertible homomorphism.

What we will now show is how to relate LG and LA structures. Here is the basic picture to keep in mind:

$$LG \underset{\text{exponentiate}}{\overset{\text{linearize}}{\rightleftarrows}} LA$$

[3]In physics literature it is common to write $X = \mathbb{1} + ix$ with x Hermitian. We will follow the standard mathematics convention.

linearization from LG to LA

From the definition of a LG, we know that we have a diffeomorphism for any $X \in G$

$$\Psi_X : G \to G \ ,$$
$$\Psi_X : Y \mapsto XYX^{-1} \ .$$

By construction, Ψ_X fixes the identity element $e \in G$. Since it is a diffeomorphism, it induces an isomorphism on the tangent space $\mathrm{Ad}_X : T_eG \to T_eG$, i.e. $\mathrm{Ad}_X \in \mathrm{Aut}(T_eG)$.[4] Note that Ad is a representation of the G, where the representation space is simply T_eG.

Definition 2.4. The adjoint representation of a LG G is the map $G \to \mathrm{Aut}(T_eG)$ given by $\mathrm{Ad} : X \to \mathrm{Ad}_X$.

Since Ad is a differentiable map of manifolds, we can take its differential at $X = e$. This yields the map

$$\mathrm{ad} : T_eG \to T(\mathrm{Aut}\, T_eG) \simeq \mathrm{End}(T_eG) \ .$$

Equivalently, ad defines a map $T_eG \otimes T_eG \to T_eG$, which we will write as $\mathrm{ad}_x(y) \in T_eG$.

A little differential geometry implies the following result:

Lemma 2.5. *The map* $\mathrm{ad}_x(y)$ *is a Lie bracket.*

Hence, given a LG G we can always construct an associated LA with vector space $\mathfrak{g} \simeq T_eG$ and the Lie bracket given by ad.

Proof sketch. Let $L_X : G \to G$ be the LG diffeomorphism $Y \mapsto XY$, i.e. multiplication from the left. For any $x \in \mathfrak{g} \simeq T_eG$, we define a vector field V_x by $V_x(X) = L'_X(x)$. Note that V_x is linear in x. This is a left-invariant vector field, in the sense that $L'_Y V_x(X) = V_x(YX)$ for all Y, and in fact all left-invariant vector fields on G arise in this way. We can use these to trivialize the tangent bundle, i.e. $TG \simeq G \times \mathfrak{g}$, as well as parametrize infinitesimal diffeomorphisms as generated by the Lie derivative $\mathcal{L}_{V_x}$.

It is a standard result that the Lie derivative is skew-symmetric on vector fields, i.e. for any two vector fields V and U on a manifold M we have $\mathcal{L}_V U = [V, U]$, where the right-hand-side is the commutator vector field. The commutator of vector fields satisfies the Jacobi identity, which is neatly summarized as $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}$.[5]

We can now apply this to defining the bracket on the Lie algebra. Using $x, y \in \mathfrak{g} = T_eG$, construct the left-invariant vector fields V_x and V_y. Then $\mathcal{L}_{V_x}V_y$ is skew-symmetric in x, y, and furthermore $[V_x, V_y]$ is a left-invariant vector field. Hence, there exists some $\ell(x, y) \in \mathfrak{g}$ such that $\mathcal{L}_{V_x}V_y = V_{\ell(x,y)}$. We can define the Lie bracket as $\ell(x, y)$, and it is not too hard to show that this definition is exactly what we obtain by our "construction" of $\mathrm{ad}_x(y)$ above.

This is too quick and too ham-fisted. For a nice treatment take a look at Helgason's book, for instance. □

[4] Recall that given any vector space V, we denote the vector space of linear maps $V \to V$ by $\mathrm{End}(V)$. Clearly $\mathrm{End}(V) \simeq V \otimes V^\vee$, where $V^\vee$ is the dual vector space. $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ is an open subset of invertible linear maps.

[5] This can be remembered by saying that the commutator of two infinitesimal diffeomorphisms is indeed a diffeomorphism, i.e. infinitesimal diffeomorphisms do form an infinite-dimensional LA.

To summarize, to obtain a LA from a LG, we set $\mathfrak{g} = T_e G$ and take the Lie bracket as defined by $\mathrm{ad} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

linearization for matrix LGs

The preceding presentation was probably incomprehensible for those unfamiliar with the mentioned aspects of differential geometry and completely boring to the initiated. Fortunately, in the case of matrix LGs, we can make all of these constructions very concrete and thereby dispel some of the mysteries and confusions. The basic point is that the differential geometry machinery means that the structures that arise in matrix LGs are present more generally. So, let's now run through the linearization program for matrix groups.

1. We start with $G \subset \mathrm{GL}(n)$ and take $X, Y \subset G$.

2. The diffeomorphism Ψ_X is then just conjugation: $\Psi_X : Y \mapsto XYX^{-1}$, and linearizing $Y = \mathbb{1} + y$, we obtain $\mathrm{Ad}_X : y \mapsto XyX^{-1}$.

3. Finally, linearizing $X = \mathbb{1} + tx + O(t^2)$, we obtain $\mathrm{ad}_x(y) = \frac{d}{dt} \mathrm{Ad}_X(y)\big|_{t=0} = [x, y]$, where the right-hand-side is just the matrix commutator.

So, linearization for matrix LGs amounts to expanding all the matrices near $\mathbb{1} \in G$ to get a vector space $\mathfrak{g}$ of linearized elements of G and observing that for any $x, y \in \mathfrak{g}$ the matrix commutator $[x, y]$ is indeed in $\mathfrak{g}$.

LA determining LG

The LA structure comes close to determining the full LG. To be precise, we have the following theorem. For a proof see Fulton and Harris, chapter 8.

Theorem 2.6. *Let G and H be LGs with G connected and simply connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be corresponding LAs. A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is the differential of a LG homomorphism if and only if φ is a LA homomorphism.*

We also have the statement that every finite LA is the LA of some LG.

Theorem 2.7. *For every LA $\mathfrak{g}$ there exists a unique, up to LG isomorphism, connected and simply connected LG G whose LA is $\mathfrak{g}$.*

The proof of this is surprisingly tricky and can be thought of in two steps:

1. the tricky step: (Ado's theorem) every $\mathfrak{g}$ can be embedded in $\mathfrak{gl}(n, \mathbb{R})$;

2. the "easy" step: Baker-Campbell-Hausdorff formula allows us to exponentiate $\mathfrak{g}$ to get a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Exponentiation

Let $X \in \mathfrak{gl}(n, \mathbb{R})$.[6] Then

$$e^X = \exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

converges to a matrix in $\mathrm{GL}(n, \mathbb{R})$, with inverse simply e^{-X}.

The Baker-Campbell-Hausdorff formula is the statement that given $x, y \in \mathfrak{gl}$ with sufficiently small norms, there exists $z \in \mathfrak{gl}$ such that $e^z = e^x e^y$. In fact, the result is constructive, since we are actually given a form for Z:

$$z = y + \int_0^1 dt \quad g(e^{t \,\mathrm{ad}_x} e^{\mathrm{ad}_y}) \cdot x \ , \tag{1}$$

where $g(z) = \log z/(z-1)$ is defined by its power series around $z = 1$.[7]

Exercise 2.8. Evaluate the first few terms in the BCH formula. You should obtain

$$z = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] + \ldots .$$

For our purposes, the important feature of the BCH formula is that every term in the power series for z only involves matrix commutators, so that if $x, y \in \mathfrak{g} \subset \mathfrak{gl}$, then the resulting z will also belong to $\mathfrak{g}$.[8] Hence, we see that indeed, as long as $\mathfrak{g}$ can be embedded in $\mathfrak{gl}(n, \mathbb{R})$, we have a nice way to construct a corresponding $G \subset \mathrm{GL}(n, \mathbb{R})$.

In fact, the BCH formula tells us that the exponential map is an isomorphism from a small neighborhood $\widetilde{U} \subset \mathfrak{g}$ containing 0 to a small neighborhood $U \subset G$ containing the identity element. This is remarkably powerful, because in a rather precise sense any small open neighborhood $U \subset G$ with $e \in \widetilde{U}$ generates G^0, the connected component of G containing the identity, by power series.[9]

Theorem 2.9. *Let U be an open neighborhood of the identity in G^0. Set $U^1 = U$ and let $U^{k+1} = UU^k$, i.e. U^{k+1} consists of all elements XY, with $X \in U^k$ and $Y \in U$. Then $G^0 = \bigcup_{k>0} U^k$.*

So, in this sense, the exponential map determines $G^0 \subset G$ from $\mathfrak{g}$: we use exp to build an open neighborhood, and then, taking power series, we obtain the full G^0.

It is important to keep in mind what the exp map does not do. Casual experience might suggest that $\exp : \mathfrak{g} \to G$ is a surjective map; a second's thought shows that at best $\exp : \mathfrak{g} \to G^0$ is surjective, and indeed, there is a key differential geometry result, the Hopf-Rinow theorem, which implies that $\exp : \mathfrak{g} \to G$ is surjective for any connected compact G. Sadly, the result is not true in general, and the counter-example is examined in the next exercise.

[6] Recall, this is a very fancy way of saying that X is an $n \times n$ matrix.

[7] These days, much valuable information on BCH may be found on the Wikipedia page.

[8] Sebastian has raised an interesting question: is it possible that while every term in the series belongs to $\mathfrak{g}$ the limit fails to do so? I believe the answer is no, essentially because $\mathfrak{g}$ is just a vector space, and hence is topologically trivial.

[9] See Bryant's lecture notes for a proof of this theorem.

Exercise 2.10. Construct $\mathfrak{sl}(2,\mathbb{R})$, the Lie Algebra corresponding to $\mathrm{SL}(2,\mathbb{R})$. Show that exp is not surjective. This is a little exercise in 2×2 matrices. Hints: first show that every $y \in \mathfrak{sl}(2,\mathbb{R})$ can be written as XmX^{-1}, where m is one of

$$\begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} , \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} , \lambda > 0 \ .$$

Use this to show a bound on $\operatorname{tr} e^y$ that is violated by more general elements of $\mathrm{SL}(2,\mathbb{R})$.

abstract exponentiation

In our linearization program we began with a definition of ad and hence $G \to \mathfrak{g}$ that applied to all LGs and not just matrix LGs. In the same spirit, one can define $\exp : \mathfrak{g} \to G$ more abstractly as the unique map such that

1. $\exp(0) = e$;
2. $\exp'(0) : T_eG \to T_eG$ is an isomorphism;
3. the image of lines through the origin of $\mathfrak{g}$ are one parameter subgroups of G, i.e. LG homomorphisms $(\mathbb{R}, +) \to G$.

Even in this more general context one can show the commutativity of the following diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\varphi]{} & H \end{array}$$

Which in particular tells us that if G is a LG with LA $\mathfrak{g}$, and $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, then $\exp(\mathfrak{h})$ is an immersed subgroup H with $T_eH = \mathfrak{h}$.

Representations

We end this lecture by describing the connection between representations of LGs and corresponding LAs. First, we make a definition.

Definition 2.11. A representation of a LA $\mathfrak{g}$ on a vector space V is a LA homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(V)$.

But now, comparing with definition 1.6, we see that a representation of a LG induces a unique representation of the corresponding LA. Conversely, given a representation V of a LA, we can reconstruct the action of G on V for a small neighborhood $U \subset G$ containing the identity, and hence, by power series of the full connected component of G. In fact, if G is connected and simply connected, then the representations of G are in 1:1 correspondence with representations of the corresponding LA.

We hope that by now there is more than enough motivation to study LAs as a way of getting a handle on LGs and their representations. We turn to LAs next.

3 The taxonomy of Lie algebras

In this section we give a first rough division of LAs into various types. For us, this will be a process of narrowing down on the structures of main interest: complex semisimple LAs.

Given a LA $\mathfrak{g}$, we have a natural concept of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$: this is a vector subspace that preserves the bracket, i.e. $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.[10]

Example 3.1. The center of a LA $Z(g)$ is a nice example of a subalgebra:

$$Z(g) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\} .$$

This is of course very analogous to the center of a group: it is the set of elements that commutes with all the other elements.

Exercise 3.2. This exercise introduces another subalgebra; this one plays a very important role in many physics applications. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Use the Jacobi identity to show that the centralizer of $\mathfrak{h} \subset \mathfrak{g}$ (also known as the commutant subalgebra), defined by

$$C_{\mathfrak{h}}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{h}\}$$

is a subalgebra.

Given a vector space V and a sub-space $W \subset V$, we can always form the quotient space V/W by imposing an equivalence relation $v \sim v + w$ for all $v \in V$ and all $w \in W$, and it is often useful to do so. We might try the same with a subalgebra as a way of decomposing $\mathfrak{g}$ into simpler components, but the resulting quotient vector space will usually not respect the bracket. However, there is a related structure that does allow for a nice quotient.

Definition 3.3. A LA ideal $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace such that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Exercise 3.4. Show that if $\mathfrak{h} \subset \mathfrak{g}$ is ideal then $\mathfrak{g} / \mathfrak{h}$ is a subalgebra.

Note that while every ideal is a subalgebra, not every subalgebra is an ideal. Every LA $\mathfrak{g}$ has the "improper" ideals 0 and $\mathfrak{g}$; the center $Z(\mathfrak{g})$ is an ideal. In addition to these, we can construct more ideals by using the bracket as follows.

1. The lower central series $\mathcal{D}_k \mathfrak{g}$ is defined recursively for $k > 0$:
   $\mathcal{D}_1 = [\mathfrak{g}, \mathfrak{g}]$, while $\mathcal{D}_{k+1} = [\mathfrak{g}, \mathcal{D}_k \mathfrak{g}]$.
2. The derived series $\mathcal{D}^k \mathfrak{g}$ is defined by $\mathcal{D}^1 \mathfrak{g} = \mathcal{D}_1 \mathfrak{g}$ and $\mathcal{D}^{k+1} \mathfrak{g} = [\mathcal{D}^k \mathfrak{g}, \mathcal{D}^k \mathfrak{g}]$.

Exercise 3.5. Show that $\mathcal{D}^k \mathfrak{g}$ is ideal and $\mathcal{D}^k \mathfrak{g} \subseteq \mathcal{D}_k \mathfrak{g}$.[11]

[10] This is a common short-hand notation that we will use quite a bit in the sequel; the long-winded form is that the bracket of any two elements in $\mathfrak{h}$ is another element in $\mathfrak{h}$.

[11] Hint: use Jacobi and induction on k.

We now come to some key definitions for sorting LAs into ridiculously simple, annoying, and very useful.

Definition 3.6. A LA $\mathfrak{g}$ is said to be

1. *abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$, i.e. $\mathfrak{g} = Z(\mathfrak{g})$;
2. *nilpotent* if $\mathcal{D}_k \, \mathfrak{g} = 0$ for some k (in particular abelian $\implies$ nilpotent);
3. *solvable* if $\mathcal{D}^k \, \mathfrak{g} = 0$ for some k (nilpotent $\implies$ solvable — see exercise (3.5));
4. *semisimple* if $\mathfrak{g}$ has no non-zero solvable ideals $\iff$ $\mathfrak{g}$ has no non-trivial abelian ideals;
5. *simple* if $\mathfrak{g}$ has no proper ideals and $\dim \mathfrak{g} > 1$.

Let's give some examples of these:

Example 3.7.

1. abelian: $\mathfrak{so}(2)$
2. nilpotent: LA of strictly upper triangular matrices;
3. solvable: LA of upper triangular matrices;
4. semisimple: $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$;
5. simple: $\mathfrak{so}(3)$.

The nilpotent and solvable cases are perhaps a little less familiar to physics folks (though think about the Heisenberg algebra), but important and non-trivial theorems of Engel and Lie state that every representation of a solvable LA has an upper triangular matrix form, so that the examples just quoted are really a very good model to keep in mind.

Given any LA $\mathfrak{g}$, we can always find the maximal solvable ideal, denoted $\mathrm{Rad}(\mathfrak{g})$. We then have a short exact sequence of LAs:

$$0 \longrightarrow \mathrm{Rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \,/\, \mathrm{Rad}(\mathfrak{g}) \longrightarrow 0 \quad .$$

Of course by construction $\mathfrak{g} \,/\, \mathrm{Rad}(\mathfrak{g})$ is semisimple. Moreover, we have Levi's theorem, which states that for complex LAs this sequence splits, i.e. there is a decomposition $\mathfrak{g} = \mathrm{Rad}(\mathfrak{g}) \oplus \mathfrak{g} \,/\, \mathrm{Rad}(\mathfrak{g})$. For LAs over reals we must use a more sophisticated decomposition, but at any rate, it is always a good idea to study the semisimple piece and its representations first.[12] We end this taxonomic tour with a definition of a frequently occurring class of LAs in physics.

Definition 3.8. A LA is reductive if $\mathrm{Rad}(\mathfrak{g}) = Z(g)$, i.e. $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{g}_{\text{abelian}}$

[12] A nice case to keep in mind is the Poincaré algebra, where it is very useful to study the Lorentz subalgebra.

The virtues of semi simplicity

Definition 3.9. A LA representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is said to be faithful if $\ker \rho = 0$.

We will be speaking of representations quite a bit, so we should bear in mind a standard abuse of terminology, where one uses interchangeably ρ and its representation space V.

Every LA $\mathfrak{g}$ has the adjoint representation that we defined in the previous lecture. In this case the vector space $V = \mathfrak{g}$, and $\rho = \mathrm{ad}$ acts via $\mathrm{ad} : x \mapsto [x, \bullet]$. That is, for any $x \in \mathfrak{g}$, $\rho(x) = \mathrm{ad}_x$ is the linear map $V \to V$ that sends a vector $y \in V = \mathfrak{g}$ to $[x, y, \in]\ \mathfrak{g}$. In general the adjoint representation need not be faithful — consider for example any abelian LA. However, since a semisimple LA has trivial center, it follows that $\mathrm{ad} : \mathfrak{g}_{ss} \to \mathfrak{gl}(\mathfrak{g}_s\, s)$ is indeed faithful. Hence, Ado's theorem, which is surprisingly tricky to prove in general, is trivial for semisimple LAs — see the discussion around theorem 2.7.

Another probably familiar definition involving representations is a notion of irreducibility.

Definition 3.10. A LA representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is irreducible if V has no $\mathfrak{g}$–invariant subspace.

This is very much parallel to irreducible representations of LGs or finite groups.

We now quote an important theorem — the details may be found in Fulton & Harris.

Theorem 3.11. *Every finite dimensional representation of a semisimple LA is completely reducible, i.e. given a rep V with an invariant subspace W, there exists a complementary invariant subspace W' such that $V = W \oplus W'$.*

This basic property fails spectacularly for non-semisimple LAs.

Another familiar notion involves the preservation of the Jordan decomposition. Recall that any $X \in \mathrm{End}(V)$ for a complex vector space V has a unique decomposition as $X = X_{ss} + X_{\mathrm{nil}}$, where X_{ss} is diagonalizable[13], while X_{nil} is nilpotent. We then have another result described in Fulton&Harris:

Theorem 3.12. *Let $\mathfrak{g}$ be a complex semisimple LA. Every $x \in \mathfrak{g}$ has a decomposition $x = x_{ss} + x_{nil}$ such that for any representation ρ $\rho(x_{ss}) = \rho(x)_{ss}$ and $\rho(x_{nil}) = \rho(x)_{nil}$. In particular, if x is diagonalizable in any faithful representation (like the adjoint representation), it is diagonalizable in all representations.*

This is another nice property that need not hold for non-semisimple LAs.

Finally, we can state the nicest thing about semisimple LAs. Each of them is a direct sum of simple LAs, and simple LAs are classified. The classification is especially simple for complex simple LAs. They are:

$$\mathfrak{sl}(n,\mathbb{C})\ , \qquad \mathfrak{so}(n,\mathbb{C})\ , \qquad \mathfrak{sp}(n,\mathbb{C})\ , \qquad \mathfrak{e}_6\ , \qquad \mathfrak{e}_7\ , \qquad \mathfrak{e}_8\ , \qquad \mathfrak{g}_2\ , \qquad \mathfrak{f}_4\ . \tag{2}$$

Thus, there are just three infinite series, already familiar from our previous constructions, as well as five exceptional cases. In what follows, we will explore these LAs, both their structure and their representations.

[13] In other words a semisimple operator.

A note on complexification

Given a real vector space V, we can always construct its complexified version by taking $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Clearly, $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V_{\mathbb{C}}$. Similarly, starting with a real LA $\mathfrak{g}_0$, we can always complexity it to obtain $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \times \mathbb{C}$. The resulting complex LA $\mathfrak{g}$ will be semisimple if and only if $\mathfrak{g}_0$ was semisimple. We will sometimes denote a complexified LA by a subscript, as in $\mathfrak{g}_{\mathbb{C}}$.

Example 3.13. Let's give a hands-on example of the complexification story. We start with $\mathfrak{su}(n)$ given in its defining representation: $n \times n$ anti-Hermitian traceless matrices.[14] We can fix a basis for these matrices, say T_A, with $A = 1, \ldots, n^2 - 1$. Any element in $\mathfrak{su}(n)$ can then be written as $x = \sum_A x^A T_A$, for real coefficients x^A. To complexity, we just declare the $x^A \in \mathbb{C}$. A moment's thought shows that the resulting set of matrices consists of traceless $n \times n$ complex matrices, which is just the LA of $\mathrm{SL}(n, \mathbb{C})$. In other words, $\mathfrak{su}(n)_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$. Similarly, $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$.

Representation tricks 1: Syms and wedges

In this section we will take a break from our gradual development of the subject and introduce some concepts that are of particular use in thinking about representations of LAs and LGs. Most of this is pretty simple linear algebra story, but perhaps it is good to have it made explicit. We will provide a few of the most useful results; more details can be found in the appendix of Fulton&Harris. To be concrete we work with vector spaces over $\mathbb{C}$.

Given two vector spaces V and W, we are all familiar with two simple ways to build new vector spaces: the direct sum $V \oplus W$ and direct product $V \otimes W$. If $\{e_i\}$ and $\{f_j\}$ are bases for V and W, then the bases for these two spaces are, respectively, $\{e_i, f_j\}$ and $\{e_i \otimes f_j\}$. Evidently,

$$\dim(V \oplus W) = \dim V + \dim W \ , \qquad \dim(V \otimes W) = \dim V \dim W \ .$$

A particularly simple vector space one often wants to study is $V \otimes V$. Just as everyone is familiar with the idea that a matrix can be written as a sum of an anti-symmetric and symmetric terms, so $V \otimes V$ can be decomposed into its symmetric and anti-symmetric components:

$$V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V \ .$$

The two components has bases $\{e_i \otimes e_j\}_{i \geq j}$ and $\{e_i \otimes e_j\}_{i > j}$; if $\dim V = n$ then $\dim \mathrm{Sym}^2 V = n(n+1)/2$ and $\dim \wedge^2 V = n(n-1)/2$. Note that if V is a representation of a LA, then $V \otimes V$ is never irreducible: the least we can do is to split it up into the symmetric and anti-symmetric components.

Sometimes it is useful to consider higher tensor products of a vector space:

$$V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{k times}} \ ,$$

and of course we then naturally consider $\mathrm{Sym}^k V$ and $\wedge^k V$, which are, respectively, completely symmetric and completely anti-symmetric tensor products of k copies of V. These are subspaces of $V^{\otimes k}$, but their direct sum does not in general span the total space.

[14] Traceless-ness follows from the nice formula $\det e^x = e^{\mathrm{tr}\, x}$.

It is often useful to think of $\mathrm{Sym}^k(V)$ as describing degree k monomials in $n = \dim V$ homogeneous coordinate x_i. For instance, this immediately tells us the dimension of the space:

$$\dim \mathrm{Sym}^k V = \binom{n+k-1}{k} . \tag{3}$$

We can construct an explicit basis as $\{e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_k}\}_{i_1 \geq i_2 \geq \cdots \geq i_k}$. Similarly, an explicit basis for $\wedge^k V$ is given by $\{e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_k}\}_{i_1 > i_2 > \cdots > i_k}$, and its dimension is $\binom{n}{k}$. The tricks of $\wedge^k V$ are probably familiar to anyone who has encountered differential forms. For instance, it is easy to see that $\wedge^{n+1} V = 0$, while $\wedge^n V = \mathbb{C}$. In addition, we have the notion of a wedge product, a bilinear map

$$\begin{aligned} \wedge &: \wedge^k V \otimes \wedge^m V \to \wedge^{k+m} V \ , \\ \wedge &: v_{i_1 \cdots i_k} \otimes w_{i_1 \cdots i_m} \mapsto v_{[i_1 \cdots i_k} w_{i_1 \cdots i_m]} \ , \end{aligned} \tag{4}$$

where we used the usual physics anti-symmetrization notation $[\cdots]$.

It is pretty easy to see that the map $\wedge : \wedge^{n-1} V \otimes V \to \mathbb{C}$ is non-degenerate, which means $\wedge^{n-1} V$ is canonically isomorphic to the dual vector space $V^\vee$, i.e. the set of all linear maps $V \to \mathbb{C}$. More generally, $\wedge^k V$ is dual to $\wedge^{n-k} V$.

Nice sum and wedge identities

$$\begin{aligned} \wedge^k (V \oplus W) &= \bigoplus_{s=0}^{k} \wedge^s V \otimes \wedge^{s-k} W \ , \\ \mathrm{Sym}^k (V \oplus W) &= \bigoplus_{s=0}^{k} \mathrm{Sym}^s V \otimes \mathrm{Sym}^{s-k} W \ , \\ \mathrm{Sym}^2 (V \otimes W) &= \mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 W \oplus \wedge^2 V \otimes \wedge^2 W \ , \\ \wedge^2 (V \otimes W) &= \wedge^2 V \otimes \mathrm{Sym}^2 W \oplus \mathrm{Sym}^2 V \otimes \wedge^2 W \ . \end{aligned}$$

Application to fundamental representation of $\mathfrak{so}(D)$

Let's apply some of these ideas to a little representation theory problem. We all know how to define $\mathfrak{so}(D)$ as the set of anti-symmetric $D \times D$ matrices — this known as the defining or fundamental representation; let's call it V. It is clearly irreducible. On the other hand, since $\mathfrak{so}(D)$ preserves a symmetric bilinear tensor δ, we know that

$$\mathrm{Sym}^2 V = \mathbf{1} \oplus W \ ,$$

where $\mathbf{1}$ denotes the trivial one-dimensional representation, while W is another (in this case irreducible) component. Now $\wedge^2 V$ is also an irreducible representation, and in fact it is isomorphic to the adjoint representation — the generators of $\mathfrak{so}(D)$ can be thought of as defined to be anti-symmetric matrices, which is of course exactly what $\wedge^2 V$ describes.[15] We can study further representations, for instance $\mathrm{Sym}^3 V$. This also reducible, because we can use the invariant tensor δ to remove a trace. In fact, that's all we can do, and $\mathrm{Sym}^3 V = V \oplus \widetilde{W}$, a sum of two irreducible representations.

OK, now let us do an example of what is known as "branching." Clearly $\mathfrak{so}(D)$ contains an $\mathfrak{so}(D-1)$ subalgebra, where we set the first column and row of the $\mathfrak{so}(D)$ matrices to 0. Under this smaller action our fundamental representation is no longer irreducible. In fact, it decomposes as

$$V = \mathbf{1} \oplus V' \ ,$$

a sum of a trivial one-dimensional representation consisting of the vector $v = (1, 0, \ldots, 0)^T$, and the $(D-1)$–dimensional fundamental representation of $\mathfrak{so}(D-1)$. Can we determine how some more complicated representation like $\widetilde{W}$ branches into irreps of $\mathfrak{so}(D-1)$? Yes, easily by using our little syms and wedges from above. For instance, we compute

$$\begin{aligned} \mathrm{Sym}^3 V &= \widetilde{W} \oplus V = \widetilde{W} \oplus V' \oplus \mathbf{1} \ , \\ \mathrm{Sym}^3(V' \oplus \mathbf{1}) &= \underbrace{V' \oplus \widetilde{W}'}_{\mathrm{Sym}^3 V'} \oplus \underbrace{W' \oplus \mathbf{1}}_{\mathrm{Sym}^2 V'} \oplus V' \oplus \mathbf{1} \ . \end{aligned} \tag{5}$$

But, since the two expressions are exactly the same, we can identify

$$\widetilde{W} = \widetilde{W}' \oplus W' \oplus V' \oplus \mathbf{1} \ .$$

Exercise 3.14. Check that all the dimensions work out, and use the same tricks to work out $\mathrm{Sym}^4 V$ and its branching into irreducible representations of $\mathfrak{so}(D-1)$.

A bit more on invariant tensors and $\wedge^k V$

Let's end with a few comments on $\wedge^k V$ for $\mathfrak{so}(D)$. The first comment is that all of these are irreducible except when $D = 2d$ and $k = d$. This has to do with the invariant tensors of $\mathfrak{so}(D)$. We will discuss this in more detail later, but at this point we can at least state the basic idea. Consider arbitrary tensor powers of the fundamental representation $V^{\otimes n}$. When we decompose these into irreps, we might find some trivial representations. A concrete example of this is given by the invariant symmetric bilinear of $\mathfrak{so}(D)$: a map $Q : \mathrm{Sym}^2 V \to \mathbb{C}$, which we can write very concretely as

$$Q(v, w) = v^i w^j \delta_{ij} = v^T w \ . \tag{6}$$

The way that we defined $\mathfrak{so}(D)$ ensures that this beast is $\mathfrak{so}(D)$–invariant. First consider the LG action of $\mathrm{SO}(D)$. By definition of the fundamental representation, this sends $v \mapsto Mv$, where M is a special orthogonal matrix. By construction $Q(Mv, Mw) = Q(v, w)$. Now when we linearize the action of M as $M = 1 + x$, where x is now an element of $\mathfrak{so}(d)$, we have the corresponding

[15] Don't worry, we will say all of this more carefully later; for now hopefully it is sensible enough to get an idea of how some of the tricks work.

statement

$$Q(xv, w) + Q(v, xw) = 0 \qquad \text{for all } x \in \mathfrak{so}(D) \text{ and } v, w \in V$$

This is what we mean by saying Q is an invariant tensor associated to representation V of $\mathfrak{so}(D)$. What sort of tensor is it? It is a map $\mathrm{Sym}^2 V \to \mathbb{C}$, i.e. $Q \in \mathrm{Sym}^2 V^\vee$. Note that since Q is non-degenerate, we can use this to define an isomorphism $V \simeq V^\vee$ between V and its dual.

Our $\mathfrak{so}(D)$ has another invariant tensor that is familiar: this is the D-dimensional ϵ tensor. Why is that? Well, by definition

$$\epsilon(v_1, \ldots, v_D) = \epsilon_{i_1 \cdots i_D} v_1^{i_1} v_2^{i_2} \cdots v_D^{i_D} \ ,$$

so it follows that under $\mathrm{SO}(D)$ action we have

$$\epsilon(Mv_1, \ldots, Mv_D) = \epsilon_{i_1 \cdots i_D} M^{i_1}{}_{j_1} M^{i_2}{}_{j_2} \cdots M^{i_D}{}_{j_D} v_1^{j_1} v_2^{j_2} \cdots v_D^{j_D} = \det M \epsilon(Mv_1, \ldots, Mv_D) \ .$$

Since $\det M = 1$, we see that this is indeed invariant.

Using $Q = \delta$ and ϵ, we can construct a map of vector spaces that commutes with the action of $\mathfrak{so}(D)$. This is known as the Hodge star. Since $V \simeq V^\vee$, we have

$$\begin{aligned} * &: \wedge^k V \to \wedge^{D-k} V \ , \\ * &: v^{i_1 \cdots i_k} \mapsto (*v)^{j_1 \cdots j_{D-k}} = \frac{1}{(D-k)!} \epsilon^{j_1 \cdots j_{D-k} i_1 \cdots i_k} v_{i_1 \cdots i_k} \ . \end{aligned}$$

Note that we liberally used δ_{ij} and δ^{ij} to lower and raise indices.

Exercise 3.15. Prove that $*^2(\wedge^k V) = (-1)^{k(D-k)}(\wedge^k V)$.[16]

Now we see what's special about the case $D = 2d$ and $k = d$. In this case $*$ is a map from $\wedge^d V$ to itself, and $*^2 = (-1)^d$. Hence, we can decompose $\wedge^d V$ into self-dual and anti-self-dual components. More precisely, if d is even then every $F \in \wedge^d V$ can be written as

$$F = \tfrac{1}{2}(F + *F) + \tfrac{1}{2}(F - *F) \ ,$$

while if d is odd then

$$F = \tfrac{1}{2}(F - i * F) + \tfrac{1}{2}(F + i * F) \ ,$$

and in each case the two terms are not mixed by $\mathfrak{so}(D)$ transformations. Hence, $\wedge^d V$ is always reducible for $\mathfrak{so}(2d)$, and we write

$$\wedge^d V = (\wedge^d V)_+ \oplus (\wedge^d V)_- \ .$$

Note that the spaces can be taken to be real if d is even but are necessarily complex when d is odd.[17] Two prominent examples of this are self-dual gauge fields in Euclidean four-dimensions (instantons), and the self-dual field-strength of IIB supergravity.

[16] In Minkowski space, i.e. with $\mathfrak{so}(1, D-1)$, we pick up $(-1)^{k(D-k)+1}$ as a phase.

[17] Because of the extra sign in $*^2$ for $\mathfrak{so}(1, 2d-1)$, this is reversed with Minkowski signature.

4 The wonders of $\mathfrak{sl}_2\mathbb{C}$

In this lecture we begin our study of simple complex LAs by examining the simplest of them all. While every physics student is deeply familiar with the basic properties of $\mathfrak{su}(2)$, we will present the results in a language that will readily generalize to other LAs. So, bear with me and enjoy the "I know this already" feeling.

The many ways to arrive at $\mathfrak{sl}_2\mathbb{C}$

The reader is probably aware of the "coincidences" in the construction of classical LGs: $\mathrm{SU}(2) = \mathrm{Sp}(1)$ and $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$. From the preceding discussions it should then be clear that all of these have isomorphic LAs. There are a couple of other ways we can obtain the same structure, the simplest being to take the LA of $\mathrm{SL}(2,\mathbb{C})$. This consists of traceless complex 2×2 matrices, for which we can take a basis[18]

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,$$

which give us the Lie bracket

$$[H, X] = 2X , \qquad [H, Y] = -2Y , \qquad [X, Y] = H . \tag{7}$$

Clearly any $x \in \mathfrak{sl}(2,\mathbb{C})$ can be written as $x = aX + bY + cH$ for some unique $a, b, c \in \mathbb{C}$. As another starting point, we can begin with the LA of $\mathrm{SL}(2,\mathbb{R})$, which has exactly the same form, except that $a, b, c \in \mathbb{R}$. Taking its complexification we are of course back at $\mathfrak{sl}_2\mathbb{C}$.

Finally, let's see how the LA of SO(3) fits into the story. The LA for SO(3) consists of 3×3 anti-symmetric matrices, a basis for which is given by

$$T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \qquad T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} .$$

These have the commutator $[T_i, T_j] = \epsilon_{ijk} T_k$. If we complexify this to $\mathfrak{so}(3,\mathbb{C})$, we can take the T_i with complex coefficients to set up a new basis

$$\widetilde{H} = 2iT_3 , \qquad \widetilde{X} = -T_1 - iT_2 , \qquad \widetilde{Y} = T_1 - iT_2 .$$

It is not hard to see that $(\widetilde{H}, \widetilde{X}, \widetilde{Y})$ have the same commutation relations as (H, X, Y) given above, so that $\mathfrak{so}(3,\mathbb{C})$ is indeed isomorphic as a LA to $\mathfrak{sl}_2\mathbb{C}$.

Exercise 4.1. Show that while $\mathfrak{so}(3,\mathbb{R}) = \mathfrak{su}(2)$, $\mathfrak{so}(3,\mathbb{R}) \neq \mathfrak{sl}(2,\mathbb{R})$.

The adjoint representation of $\mathfrak{sl}_2\mathbb{C}$

So far we have presented the LA in its *defining* or *fundamental* representation — this is two-dimensional for $\mathfrak{sl}_2\mathbb{C}$ and more generally is n-dimensional for $\mathfrak{sl}_n\mathbb{C}$, of course. The adjoint represen-

[18] As we will see, this particular basis turns out to be very convenient for the study of general LAs, so it's a good idea to get comfortable with it.

tation is in a sense more natural since it is determined by the defining characteristic of the LA — the Lie bracket. What form does it take? To determine this, we use (7), as well as the definition of ad:

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \ , \\ x &\mapsto \{y \to [x,y]\} \ . \end{aligned}$$

So, let's work this out:

$$\begin{aligned} \mathrm{ad}_X &= \{aX + bY + cH \to -2cX + bH\} \ , \\ \mathrm{ad}_Y &= \{aX + bY + cH \to +2cY - aH\} \ , \\ \mathrm{ad}_H &= \{aX + bY + cH \to +2aX - 2bY\} \ . \end{aligned}$$

So, we can represent this as an action on the column vector $(a,b,c)^T$ as

$$\mathrm{ad}_X = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ , \qquad \mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \ , \qquad \mathrm{ad}_H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

We see very clearly that ad is indeed faithful and in fact irreducible, so, as expected, $\mathfrak{sl}_2\mathbb{C}$ is indeed a simple LA.

Exercise 4.2. Verify that this is indeed a representation of our algebra, i.e. $[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_H$, etc. Use the same procedure to construct the adjoint representation for $\mathfrak{so}(3)$ from the generators above.

We will have ample opportunity for studying various representations of LAs, and to that end it is good to know when two representations are the same. We will use the following definition.

Definition 4.3. Two LA representations $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ are isomorphic if we can find a linear invertible map $\varphi : V_1 \to V_2$ so that $\varphi\rho_1(x)v = \rho_2(x)\varphi v$ for all $x \in \mathfrak{g}$ and $v \in V_1$, i.e. $\rho_1(x) = \varphi^{-1}\rho_2(x)\varphi$ for all $x \in \mathfrak{g}$.

Exercise 4.4. Show that for $\mathfrak{so}(3)$ the fundamental representation is isomorphic to the adjoint representation.[19]

Is there a smaller simple LA?

One might wonder if there is some more basic example of a simple LA with dim < 3. The answer is no. Dimension 1 LAs are obviously abelian, while for dimension 2 we observe that since $[\cdot,\cdot] : \wedge^2 \mathfrak{g} \to \mathfrak{g}$, and $\dim \wedge^2 \mathfrak{g} = 1$ if $\dim \mathfrak{g} = 2$, $[\mathfrak{g},\mathfrak{g}]$ is either trivial or is an ideal in $\mathfrak{g}$. So, we are indeed studying the smallest dimension possible for a simple LA. As a fun exercise, you might want to try prove that every complex simple LA with dim $= 3$ is isomorphic to $\mathfrak{sl}_2\mathbb{C}$.

[19] More generally, we will see that $\mathrm{ad}(\mathfrak{so}(n))$ is isomorphic to $\wedge^2 V$, where V is the n-dimensional fundamental representation. Here we have the special feature of $\mathfrak{so}(3)$ that $\wedge^2 V = V^\vee = V$.

Irreducible representations of $\mathfrak{sl}_2\mathbb{C}$

We will now go over the classification of finite-dimensional irreducible representations (irreps) of our favorite LA. Let $\rho : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{gl}(W)$ be a finite-dimensional irrep. We will use a frequently employed abuse of notation, where we will simply refer to the irrep by the vector space W and write H, X, Y instead of $\rho(H), \rho(X), \rho(Y)$ as long as confusion is unlikely to arise.

The first step in our classification is to observe that due to theorem 3.12, since H is diagonalizable in the adjoint representation, it is also diagonalizable in W. Hence, we can split W into eigenspaces of H:

$$W = \bigoplus_\alpha W_\alpha \ , \qquad HW_\alpha = \alpha W_\alpha \ ,$$

where $\alpha \in \mathbb{C}$ labels the eigenvalues that occur. Moreover, there exists a maximum value of α such that $XW_{\alpha_{\max}} = 0$. This follows because $\dim W < \infty$ and $H(XW_\alpha) = (\alpha + 2)XW_\alpha$. Let us for brevity set $n = \alpha_{\max}$.

Next, we pick a non-zero vector $w \in W_n$ and construct a subspace of $U \subseteq W$ as

$$U = \operatorname{span}\{w, Yw, Y^2w, \dots\} \ .$$

In fact, $U = W$. Since W is an irrep, it is sufficient to show that $\mathfrak{sl}_2\mathbb{C}$ preserves U, and for that all we need to show is $XY^m w \in U$ for any $m \geq 0$. This is easily done by examining the terms m by m:

$$\begin{aligned}
&m = 0 && Xw = 0 \in U \ , \\
&m = 1 && XYw = [X, Y]w = nw \in U \ , \\
&m = 2 && XY^2w = XY^2w = (n-2)Yw + Y[X, Y]w = (2n-2)Yw \in U \ , \\
&\ \vdots && \\
& && XY^m w = (n - 2(m-1) + n - 2(m-2) + \cdots + n)Y^{m-1}w \\
& && \qquad = m(n - m + 1)Y^{m-1}w \in U.
\end{aligned}$$

So, indeed $U = W$, and this has a number of implications:

1. $\dim W_n = 1$ (or $W \neq U$)

2. $n \in \mathbb{Z}_{\geq 0}$ since this is an $m > 0$ such that $Y^m w = 0$ but $Y^{m-1}w \neq 0$. Thus,

$$0 = XY^m w = m(n - m + 1)Y^{m-1}w \implies n = m - 1 \ .$$

3. $\dim W = n + 1$, and we have an explicit basis

$$W = w \oplus Yw \oplus Y^2w \oplus \cdots Y^n w \ ,$$

where each term $Y^k w$ is a one-dimensional eigenspace of H with eigenvalue $n - 2k$.

We have shown that $\mathfrak{sl}_2\mathbb{C}$ irreps are labeled by a non-negative integer n, and we will use the notation Γ_n to refer to these irreps. It follows that any finite-dimensional representation can be decomposed as $V = \bigoplus_s \Gamma_{n_s}$. The physics translation of this is that Γ_n is the spin $j = n/2$ irrep. Note that the set of non-isomorphic irreps is infinite for $\mathfrak{sl}_2\mathbb{C}$. This is a general feature of LAs and LGs that should be contrasted with finite groups: every finite group has a finite number of irreps.[20]

An example of "branching"

Here is a typical problem encountered in the theory of LAs and their representations. We have a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Any representation of $\mathfrak{g}$ is then obviously a representation of $\mathfrak{h}$, and we often want to know how to decompose it into the irreps of $\mathfrak{h}$. This is known as "branching". We encountered an example of this in our discussion of $\mathfrak{so}(D)$ representations in the previous lecture, and our decomposition of Γ_n into H-eigenspaces is another (very simple) example. In particular, we think of H as generating $\mathfrak{u}(1) \subset \mathfrak{sl}_2\mathbb{C}$. All irreps of $\mathfrak{u}(1)$ are one-dimensional and labeled by the "charge," i.e. the H eigenvalue. Let's call these irreps Σ_q if $H\Sigma_q = q\Sigma_q$. So, our branching is phrased as follows:

$$\begin{aligned} &\mathfrak{sl}_2\mathbb{C} \supset \mathfrak{u}(1) \\ &\Gamma_n = \Sigma_n \oplus \Sigma_{n-2} \oplus \Sigma_{n-4} \oplus \cdots \oplus \Sigma_{-n} \ . \end{aligned}$$

The simple fact that not all irreps of $\mathfrak{h}$ arise in decompositions of $\mathfrak{g}$ representations — for instance, in our example we see that all the obtained H-eigenvalues are integral — is a key motivation for such grand ideas as grand unification.

Weights and roots

We have just encountered, in a rather simple guise, some concepts that will be central to our study of much more interesting LAs. So, let us give some names to these.

Given a representation W, not necessarily irreducible, we can always decompose it into the H eigenspaces as above: $W = \bigoplus_\alpha W_\alpha$. These eigenspaces are known as the *weight spaces* of the representation, and the values $\alpha \in \mathbb{C}$ are the weights. As we have seen, the weights determine the representation, and this will remain true for more general LAs as well. The maximum weight of an irrep, n in the above discussion, is called the *highest weight*.

If W is the adjoint representation, we have a special term for the non-zero weights and weight spaces: they are known, respectively, as roots and root spaces.[21] Note that any two weights of a representation are related by adding or subtracting roots.

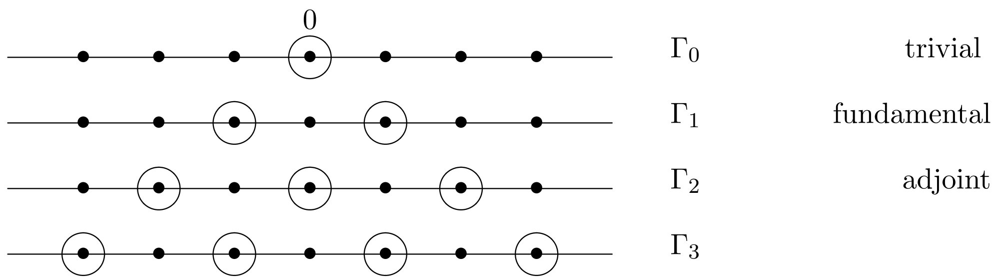
All possible weights of a LA generate a lattice, called, appropriately enough, the weight lattice and denoted by Λ_W. In our case we see $\Lambda_W \simeq \mathbb{Z}$. The roots, being particular weights, generate a sub-lattice known as the root lattice: $\Lambda_R \subset \Lambda_W$. In our case we have $\Lambda_R = 2\mathbb{Z} \subset \mathbb{Z}$, and $\Lambda_W/\Lambda_R = \mathbb{Z}_2$.

We can usefully describe various representations by giving their weight diagrams, i.e. by displaying which weights in Λ_W a particular representation happens to provide. For instance, we have

[20] Recall that for a finite group G the dimensions of the irreps Γ_i satisfy $|G| = \sum_i (\dim \Gamma_i)^2$—every irrep appears $\dim \Gamma$ times in the regular representation.

[21] The adjoint representation also has zero weights that correspond to generators of the Cartan subalgebra.

the following weight diagrams for Γ_0, Γ_1, and Γ_2:



In each case the circled lattice points are the weights of the particular irrep.

Tensoring irreps and plethysms

In addition to branching, the other typical problem we encounter in representation theory is how to decompose a representation of $\mathfrak{g}$ into its irreducible components. In particular, suppose we are smart enough to produce the full (infinite) list of distinct irreps $\{\Gamma_i\}$. We still have an interesting question: how do tensor products of the Γ_i decompose into irreps? This is known as the *Clebsch-Gordan problem*. Clearly we can reduce the question to the basic decomposition

$$\Gamma_i \otimes \Gamma_j = \bigoplus_k (\underbrace{\Gamma_k \oplus \Gamma_k \oplus \cdots \oplus \Gamma_k}_{N_{ij}^k \text{ copies}}) = \bigoplus_k \Gamma_k^{\oplus N_{ij}^k} .$$

In the last equality we used a common notation that $V^{\oplus n}$ is a direct sum of n copies of V; there is a similar short-hand $V^{\otimes n}$ which is the tensor product of n copies of V. The non-negative integer N_{ij}^k clearly satisfies $N_{ij}^k = N_{ji}^k$ and is the *multiplicity* with which representation Γ_k occurs in the decomposition of $\Gamma_i \otimes \Gamma_j$. In the lectures that follow we will learn how to describe the irreps of simple complex LAs, and then nice software packages such as LiE or LieArt are very good for computing the multiplicities — for large enough LAs and irreps, it is definitely not something to do by hand. Analytic expressions for the multiplicities are in general not available and when available quickly become cumbersome.

Sometimes one wants to know more than just the N_{ij}^k but an explicit isomorphism of vector spaces $\Gamma_i \otimes \Gamma_j$ in some preferred choice of basis. In general this can be a bit tricky, especially if one wants to work with some particular orthonormal bases for Γ_k. Fortunately, for many questions such detailed constructions are not necessary, and it is sufficient to just match up the weight spaces of the left- and right-hand sides. Indeed, for many problems it is sufficient to just know the N_{ij}^k.

A very much related notion that comes often comes up is that of *plethysm*: given a representation V, how do various derived representations, such as $V^\vee$, $\operatorname{Sym}^k V$, and $\wedge^k V$ decompose into irreps?

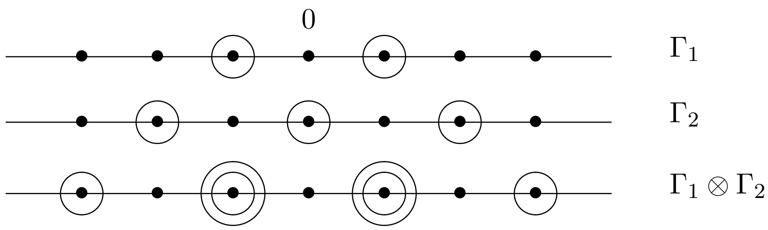
For $\mathfrak{sl}_2\mathbb{C}$ the story is quite simple and reduces to the basic Clebsch-Gordan decomposition. Let's illustrate it with a basic example. First we note that if $v \otimes w \in V \otimes W$, then the action of H on the tensor product is[22]

$$H(v \otimes w) = (Hv) \otimes w + v \otimes (Hw) \ .$$

So, in particular, decomposing $\Gamma_n = \bigoplus_\alpha (\Gamma_n)_\alpha$ and $\Gamma_m = \bigoplus_\beta (\Gamma_n)_\beta$ into the weight spaces, we see that

$$\Gamma_n \otimes \Gamma_m = \bigoplus_{\alpha,\beta} \underbrace{(\Gamma_n)_\alpha \otimes (\Gamma_n)_\beta}_{\text{weight } \alpha+\beta \text{ eigenspace}} \quad .$$

So, to obtain the Clebsch-Gordan decomposition in this weight basis, all we need to do is group the $\alpha + \beta$ weights into irreps — an easy task. Let us illustrate it with $\Gamma_1 \otimes \Gamma_2$, for which we have the following weight diagram:



The multiple circles indicate that the weights occur with multiplicity — in this case 2 for weights ± 1. To finish the decomposition, we note that the unique highest weight $+3$ is the highest weight of Γ_3, and selecting out the weights of Γ_3, we see that the left-over weights just constitute Γ_1. Hence, $\Gamma_1 \otimes \Gamma_2 = \Gamma_3 \oplus \Gamma_1$.

Exercise 4.5. Show that for $n \geq m$

$$\Gamma_n \otimes \Gamma_m = \Gamma_{n+m} \oplus \Gamma_{n+m-2} \oplus \cdots \oplus \Gamma_{n-m} \ .$$

Prove that $\Gamma_n = \mathrm{Sym}^n V$, where V is the two-dimensional fundamental representation.

[22]Note the abuse of notation here. More properly, we would denote by ρ_V and ρ_W the maps from $\mathfrak{sl}_2\mathbb{C}$ to $\mathrm{End}(V)$ and $\mathrm{End}(W)$, respectively, and then the proper statement would be

$$\rho_{V \otimes W}(H)(v \otimes w) = (\rho_V(H)v) \otimes w + v \otimes (\rho_W(H)w).$$

For obvious reasons we use the overloaded notation, but it is sometimes useful/clarifying to write out the details. Don't forget that if you get confused.

5 Structure of complex simple LAs, I

In this lecture we begin to investigate in depth the structure of simple LAs. Our study of $\mathfrak{sl}_2\mathbb{C}$ will be a key ingredient that will lead to important quantization results. We will cover the Cartan subalgebra, roots, the Killing form, and distinguished $\mathfrak{sl}_2\mathbb{C}$ subalgebras of any simple LA.

For starters, we fix a complex simple LA $\mathfrak{g}$, with $\mathfrak{g} = \mathfrak{sl}_3\mathbb{C}$ as our working example.

The Cartan subalgebra

Definition 5.1. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a maximal diagonalizable abelian subalgebra.

While its existence might be very intuitively clear, it is not completely trivial to show that it exists. Nevertheless, this is possible by completely algebraic methods, and it is also possible to show that it is unique up to $\mathfrak{g}$-automorphisms.

Example 5.2. For instance, for $\mathfrak{sl}_3\mathbb{C}$, defined by its fundamental 3-dimensional representation, we can take $\mathfrak{h}$ to consist of traceless diagonal matrices:

$$\mathfrak{h} = \left\{ H = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} , a_1 + a_2 + a_3 = 0 \right\} .$$

It is easy to convince yourself (and you should) that no other elements of $\mathfrak{sl}_3\mathbb{C}$ will commute with all of $\mathfrak{h}$, so this is indeed maximal.

While we are at it, we can also describe the remaining generators in a convenient fashion. For instance, we can take them to be 6 matrices E_{ij}, with $i \neq j$ and $1 \leq i, j \leq 3$, such that E_{ij} has a 1 in the i, j-th spot and zeroes everywhere else. You should also check that $[H, E_{ij}] = (a_i - a_j)E_{ij}$.

We now come to a key characterization of a LA:

Definition 5.3. The rank of $\mathfrak{g}$ is the dimension of its Cartan subalgebra.

Of course we see that $\mathfrak{sl}_3\mathbb{C}$ has rank 2.

The Cartan subalgebra $\mathfrak{h}$ is a vector space, and so of course has a dual $\mathfrak{h}^\vee$— this is the vector space of all linear maps from $\mathfrak{h}$ to the base field, in our case $\mathbb{C}$; a fancy way of writing this that you may encounter at some point is $\mathfrak{h}^\vee = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$.

Example 5.4. In our running example this is given by

$$\mathfrak{h}^\vee = \{(\mathbb{C}^3) \ \text{ spanned by } \ L_1, L_2, L_3\}/(L_1 + L_2 + L_3 \sim 0) \ ,$$

with $L_i(H) = a_i$. Note that of course the rank of $\mathfrak{sl}_3\mathbb{C}$ is 2, but we choose to work with $\mathbb{C}^3$ modulo an equivalence relation to keep manifest the symmetry in $a_{1,2,3}$.

The utility of $\mathfrak{h}^\vee$ is that it gives us a handy way of keeping track of the eigenvalues and eigenspaces of any representation under the action of $\mathfrak{h}$. For $\mathfrak{sl}_2\mathbb{C}$, where $\mathfrak{h}$ was one-dimensional, we could do this with a single complex number for each eigenspace; however, now we will need $\operatorname{rank}\mathfrak{g} = \dim\mathfrak{h}$ complex numbers to give a complete characterization, and these are naturally identified with elements of $\mathfrak{h}^\vee$.

Let's elaborate on this a little bit more. From our discussion of theorem 3.12 we know that $\mathfrak{h}$ will be diagonalizable in any representation V. That means that if we pick a basis H^i, $i = 1, \ldots, \operatorname{rank} \mathfrak{g}$ for $\mathfrak{h}$, then any representation decomposes as

$$V = \bigoplus_{\omega} V_\omega \ ,$$

where $H^i V_\omega = \omega^i V_\omega$, and $\omega^i \in \mathbb{C}$. More generally, there is a nice linear structure here. We can write any $H \in \mathfrak{h}$ as $H = \sum_i a_i H^i$ and then

$$HV_\omega = \textstyle\sum_i a_i \omega^i V_\omega = \omega(H) V_\omega \ .$$

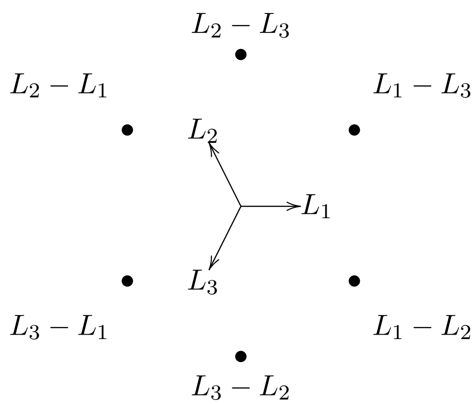
So, we see that the eigenvalues ω are canonically elements of $\mathfrak{h}^\vee$: each set of eigenvalues gives a $\mathbb{C}$-linear map from $\mathfrak{h} \to \mathbb{C}$. Just as we did for $\mathfrak{sl}_2\mathbb{C}$, we call these eigenvalues ω *weights*, and the V_ω the weight spaces. We will also occasionally refer to $\dim V_\omega$ as the *multiplicity* of the weight.

In the $\mathfrak{sl}_2\mathbb{C}$ case, we defined the roots as the weights of the adjoint representation. Now more generally, we let $\mathfrak{h}$ act on $\mathfrak{g}$ by the adjoint representation. Diagonalizing the action, we obtain the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha \ ,$$

where $\alpha \in \mathfrak{h}^\vee$ are the *roots* of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ are the root spaces. Remarkably, we will show that $\dim \mathfrak{g}_\alpha = 1$ for each root of a simple LA.

Example 5.5. Continuing the example, we saw that $[H, E_{ij}] = (a_i - a_j)E_{ij}$, so that each E_{ij} is an H–eigenspace with respect to the adjoint action, and if we work with our basis $L_{1,2,3}$ for $\mathfrak{h}^\vee$, we can write the corresponding root as $\alpha = L_i - L_j$. Since all of the roots are real, we can easily plot them by taking a real slice through $\mathfrak{h}^\vee$:



As we will see shortly, we will always be able to pick a basis for $\mathfrak{h}$ and $\mathfrak{h}^\vee$ so that all the roots are real, so that we can always take this sort of real slice.

We denote the set of all roots $\alpha \in \mathfrak{h}^\vee$ with $\mathfrak{g}_\alpha \neq 0$ by R.

Exercise 5.6. Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ for two roots $\alpha, \beta \in R$. Use Jacobi identity to show that $[x, y] \in \mathfrak{g}_{\alpha+\beta}$.

The Killing form

Any LA $\mathfrak{g}$ admits a symmetric bilinear form

$$B(\cdot,\cdot) : \mathrm{Sym}^2\,\mathfrak{g} \to \mathbb{C}\ ,$$
$$B(x,y) := \mathrm{Tr}_{\mathrm{adj}}(\mathrm{ad}_x\,\mathrm{ad}_y)\ , \tag{8}$$

This is the Killing form. Note that there is nothing special about using the adjoint representation here: we could have similarly used any representation $\rho : \mathfrak{g} \to V$ and taken $B_V(x,y) = \mathrm{Tr}_V(\rho(x)\rho(y))$.[23] However, the point about the adjoint representation is that of course every LA has it, and it is faithful for semisimple LAs.

We now describe a few key properties of B. First, B is the prototypical example of an invariant tensor. We will discuss this more intrinsically in what follows, but for now let's state the operational definition:

$$B([x,y],z) = B(x,[y,z])\ . \tag{9}$$

This is easily shown by using the definition of B and cyclicity of the trace. A more non-trivial result is the following theorem.

Theorem 5.7. *If $\mathfrak{g}$ is simple, then B is non-degenerate.*

In fact, this can be generalized.

Theorem 5.8 (Cartan's criterion)**.** *B is non-degenerate if and only if $\mathfrak{g}$ is semi-simple; $B([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = 0$ if and only if B is solvable, and B is identically 0 if and only if $\mathfrak{g}$ is nilpotent.*

Proof. Let's prove theorem 5.7. The proof has two parts. The first is quite easy: invariance of B implies that $\ker B$ is ideal in $\mathfrak{g}$. Hence, for a simple $\mathfrak{g}$ we have two possibilities: either $\ker B = 0$ or $\ker B = \mathfrak{g}$. To exclude the latter possibility we sketch a linear algebra argument from notes by Kirillov [4]. Fix any faithful representation V of $\mathfrak{g}$. Given a diagonalizable $x \in \mathfrak{gl}(V)$, there exists an element $\overline{x} \in \mathfrak{gl}(V)$, constructed as a polynomial in x, that has same eigenspaces as x but complex conjugate eigenvalues. Moreover, the commutator action $\overline{x} : y \to [\overline{x}, y]$ can then be expressed as a polynomial $Q([x,\cdot])(y)$. So, suppose we have a simple LA $\mathfrak{g}$ with $B(x,y) = 0$ for all $x, y \in \mathfrak{g}$. Let x be a non-zero diagonalizable element. Then by definition of $\overline{x}$ $\mathrm{Tr}(x\overline{x}) > 0$. On the other hand, we can write $x = [y,z]$ and then write

$$\mathrm{Tr}(x\overline{x}) = \mathrm{Tr}([y,z]\overline{x}) = -\,\mathrm{Tr}(y[\overline{x},z]) = 0\ ,$$

where in the last equality we used the fact that $[\overline{x},z] = Q([x,\cdot])(z) \in \mathfrak{g}$, and hence by assumption the trace vanishes. We have reached a contradiction, so either $\mathfrak{g}$ is not simple or $\ker B \neq \mathfrak{g}$. □

Uses of the Killing form

The non-degeneracy of the Killing form has a number of important applications for simple $\mathfrak{g}$.

[23] We will explore the relation between this multitude of Killing forms in the sequel; we will discover that all B_V for faithful representations differ by at most a constant. That constant, when properly normalized, is the Dynkin index of a representation.

1. "Orthogonality" of the root spaces. Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then for all $H \in \mathfrak{h}$

$$B(H,[x,y]) = \begin{cases} B(y,[H,x]) = \alpha(H)B(x,y) \ , \\ B(x,[y,H]) = -\beta(H)B(x,y) \ , \end{cases} \qquad \Longrightarrow \ (\alpha+\beta)(H)B(x,y) = 0 \ .$$

   Hence $B(x,y)$ vanishes if $\beta \neq -\alpha$. Note that this also holds if we think of $\beta = 0$, i.e. $\mathfrak{g}_\beta = \mathfrak{h}$ as a sort of degenerate root space[24], so that either $\alpha = 0$, i.e. $x \in \mathfrak{h}$ as well, or $B(x,y) = 0$ for all $y \in \mathfrak{h}$.

   In particular if α is a root then $-\alpha$ must be a root as well; otherwise B would be degenerate.

2. Isomorphism $\mathfrak{h}^\vee = \mathfrak{h}$. For any $\sigma \in \mathfrak{h}^\vee$ we define $T_\sigma \in \mathfrak{h}$ by

$$B(T_\sigma, H) = \sigma(H) \qquad \text{for all} \quad H \in \mathfrak{h} \ .$$

   Since $\ker B = 0$, and $B(x,H) = 0$ unless $x \in \mathfrak{h}$, this has a unique solution for T_σ.

3. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$ for any root α. This follows from the previous statement: otherwise $x, y \in \ker B$. In particular, we have $[x,y] = B(x,y)T_\alpha$.

4. $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$. To show this, one considers the algebra generated by x, y and $T_\alpha \neq 0$ we just constructed, with the extra assumption that $[T_\alpha, x] = 0$. This is a solvable LA, and by a bit more manipulation one can show that $T_\alpha \in \mathfrak{h}$ must be representable by a strictly upper-triangular matrix. But since T_α is also diagonalizable it must in fact be zero, a contradiction. So, since $[T_\alpha, x] = \alpha(T_\alpha)x$, we now know $\alpha(T_\alpha) \neq 0$.

Distinguished $\mathfrak{sl}_2\mathbb{C}$s

Using the results assembled in the previous section, we immediately see for every non-zero root $\alpha \in R$ there exist $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ such that

$$[x,y] = B(x,y)T_\alpha \ , \qquad [T_\alpha, x] = \alpha(T_\alpha)x \ , \qquad [T_\alpha, y] = -\alpha(T_\alpha)y \ .$$

Thus, we can build a canonically normalized $\mathfrak{sl}_2\mathbb{C}$ algebra by taking generators

$$H_\alpha = \frac{2}{\alpha(T_\alpha)}T_\alpha \ , \qquad X_\alpha = \sqrt{\tfrac{2}{B(x,y)\alpha(T_\alpha)}}\,x \ , \qquad Y_\alpha = \sqrt{\tfrac{2}{B(x,y)\alpha(T_\alpha)}}\,y \ .$$

We will call this algebra $\mathfrak{s}_\alpha$. Note that $\alpha(H_\alpha) = 2$.

This has a very important implication for any representation V of $\mathfrak{g}$ with weight decomposition

$$V = \bigoplus_\omega V_\omega \ ,$$

simply because V furnishes a representation of each $\mathfrak{s}_\alpha$, with H_α eigenvalue $\omega(H_\alpha)$. But, we showed in the previous lecture that H_α eigenvalues must be integral! Hence, we obtain the following theorem.

[24] This way of thinking about the Cartan subalgebra is sometimes useful.

Theorem 5.9. *Every weight ω of a representation of $\mathfrak{g}$ must satisfy $\omega(H_\alpha) \in \mathbb{Z}$ for every root α.*

Thus, the allowed weights belong to an integral lattice $\Lambda_W \in \mathfrak{h}^\vee$. This is known, sensibly enough, as the *weight lattice* of $\mathfrak{g}$.

Integrality also leads to important constraints on the possible roots. To obtain these, fix $\alpha \in R$ and consider the $\mathfrak{s}_\alpha$ representation consisting of $\mathfrak{h}$ and all $\mathfrak{g}_\beta$ where β is proportional to α. That is, we take

$$V = \mathfrak{h} \oplus \bigoplus_i \mathfrak{g}_{k_i\alpha} \ ,$$

where a priori $k_i \in \{\mathbb{C} \setminus 0\}$.

Exercise 5.10. Show that V is a representation of $\mathfrak{s}_\alpha$, with the action inherited from the adjoint representation of $\mathfrak{g}$.

As we will now argue, we in fact have the following result.

Theorem 5.11. $V = \mathfrak{h} \oplus \mathfrak{s}_\alpha$.

Proof. The first observation is that the weight space $\mathfrak{g}_{k_i\alpha}$ has weight $k_i\alpha(H_\alpha) = 2k_i$, and integrality forces $k_i \in \frac{1}{2}\mathbb{Z}$. Next, we observe that $\mathfrak{s}_\alpha$ acts trivially on $\ker(\alpha) \in \mathfrak{h}$ and irreducibly on $\mathfrak{s}_\alpha \subset V$, so that the zero weightspace, which is the full $\mathfrak{h}$ is given by $\ker(\alpha) \oplus H_\alpha$. But that implies that the only integral values of k_i are $0, \pm 2$, since otherwise we would have an extra even weight in V, which would necessarily be accompanied by an additional zero weight space. For the same reason the ± 2 weights, which are present due to $\mathfrak{s}_\alpha \subset V$, must each occur with multiplicity 1.[25]

In particular, this means that if α is a root then 2α cannot be root. It then follows that if α is a root, then $\alpha/2$ cannot be a root, which means $k_i = \pm 1/2$ is also forbidden. But this means that all other half-integral k_i are excluded as well, and the result follows. □

So, in fact, every one of our $\mathfrak{g}_\alpha$ is a one-dimensional space, and we can roughly think of the full $\mathfrak{g}$ as a collection of $\oplus_\alpha \mathfrak{s}_\alpha$ modulo relations in the Cartan subalgebra. This is already very nice, since it reduces a potentially mysterious structure of a general simple LA to a collection of $\mathfrak{sl}_2\mathbb{C}$s. In the next lecture, we will see that in fact we can reduce the story further to just rank $\mathfrak{g}$ *simple roots*. The structure of these will be encoded in the Dynkin diagram, and it will allow us to describe all complex simple LAs and their irreps in a remarkably complete fashion.

[25] If this is still confusing take a look at the weight diagrams for irreps of $\mathfrak{sl}_2\mathbb{C}$ from the previous lecture: every Γ_{2m} necessarily has a 0 weight, and every Γ_{2m+1} irrep necessarily has weights ± 1.

6 Structure of complex simple LAs, II

Let's give a little summary of our achievements so far. We have argued/shown that a simple LA $\mathfrak{g}$ has

1. a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with dual vector space $\mathfrak{h}^\vee$; $\dim \mathfrak{h}$ is the rank of $\mathfrak{g}$;
2. a non-degenerate Killing form $B(x, y) = B(y, x)$;
3. root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} g_\alpha$, where the roots generate the root lattice Λ_R, a sub lattice of $\Lambda_W \subset \mathfrak{h}^\vee$;
4. distinguished $\mathfrak{sl}_2\mathbb{C}$s, $\mathfrak{s}_\alpha$ for every root α, with a corresponding distinguished $H_\alpha \in \mathfrak{h}$;
5. a decomposition of any rep into weight spaces $V = \oplus_\omega V_\omega$, with $\omega(H_\alpha) \in \mathbb{Z}$ for all weights ω and all roots α.

Today we will pursue some implications of these facts, as well as introduce a few more ingredients that will help us to characterize all finite-dimensional representations of $\mathfrak{g}$.

Some further observations

We will make a few simple observations that follow from the properties we just reviewed. To start, we note that

$$\dim \mathfrak{g} = \operatorname{rank} \mathfrak{g} + \underbrace{\#(\text{roots})}_{\in 2\mathbb{Z}} \ .$$

For instance, $\mathfrak{sl}_n\mathbb{C}$ has $\dim = n^2 - 1$, rank $= n - 1$, and $n(n-1)$ roots. Next, it should be clear that the H_α span $\mathfrak{h}$, since otherwise there would be some H' with a trivial action on all roots, i.e. we would have $\mathrm{ad}_{H'} = 0$. That is impossible for a simple LA.

Spaces and lattices

The roots generate an integral lattice in $\mathfrak{h}^\vee$, Λ_R, a subset of Λ_W, the weight lattice. While both of these lattices are naturally embedded in $\mathfrak{h}^\vee$, a complex vector space, it is very convenient to consider them as embedded in a real subspace of $\mathfrak{h}^\vee$, defined by $\Lambda_R \otimes \mathbb{R} \simeq \Lambda_W \otimes \mathbb{R} \simeq \mathbb{R}^{\dim \mathfrak{h}} \subset \mathfrak{h}^\vee$. Note that since Λ_R is a sublattice of Λ_W, they yield the same real vector space when tensored with $\mathbb{R}$. Let us call this real vector space $\mathbb{E}$.[26] The H_α then belong to a dual real vector space $\mathbb{E}^\vee \subset \mathfrak{h}$.

Lemma 6.1. *B when pulled back to $\mathfrak{h}$ gives a positive-definite form, i.e. a metric on $\mathbb{E}^\vee$.*

Proof. To see this, note that $B(H, x) = 0$ for any $H \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$.[27] That means B must be non-degenerate on $\mathfrak{h}$, since otherwise some non-zero $H \in \mathfrak{h}$ would be in $\ker B$. On the other hand, we have for any $H, \widetilde{H} \in \mathfrak{h}$

$$B(H, \widetilde{H}) = \sum_{\gamma \in R} \gamma(H)\gamma(\widetilde{H}) \ ,$$

[26] $\mathbb{E}$ for Euclid.

[27] The argument is analogous to the "orthogonality" statement from the previous lecture.

and in particular for $\mathbb{E}^\vee$, which is spanned by the H_α, we have

$$B(H_\alpha, H_\beta) = \sum_{\gamma \in R} \gamma(H_\alpha)\gamma(H_\beta) \ ,$$

a manifestly positive-definite form since $\gamma(H_\alpha) \in \mathbb{Z}$. □

Of course once we have a metric on a vector space, we also naturally get a metric on the dual space by using the isomorphism $\mathfrak{h} \to \mathfrak{h}^\vee$ $\alpha \mapsto T_\alpha$, defined by $B(T_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$.[28] The resulting metric $b : \mathrm{Sym}^2 \to \mathbb{R}$ is

$$b(\alpha, \beta) = B(T_\alpha, T_\beta) = \alpha(T_\beta) = \beta(T_\alpha) \ .$$

Since for any weight β we have $\beta(H_\alpha) \in \mathbb{Z}$ for all roots, it follows that

$$\beta(H_\alpha) = \frac{2}{\alpha(T_\alpha)} B(T_\alpha, T_\beta) = \frac{2b(\beta,\alpha)}{b(\alpha,\alpha)} \ \in \mathbb{Z} \tag{10}$$

for any weight β and any root α.

The Weyl group

Having completed some simple elaborations of preceding results, we now turn to something new: an important discrete symmetry group of the roots and weights. For any $\alpha \in R$ we define an isomorphism $W_\alpha : \mathfrak{h}^\vee \to \mathfrak{h}^\vee$ by

$$W_\alpha(\beta) = \beta - \frac{2b(\alpha,\beta)}{b(\alpha,\alpha)}\alpha \ .$$

Observe that β is left invariant if and only if $b(\alpha, \beta) = 0$; $W_\alpha(\alpha) = -\alpha$, and W_α^2 is the identity. Evidently W_α acts on $\mathbb{E}$ as a reflection through the plane orthogonal to the root α. The Weyl group $\mathcal{W}$ is the finite group generated by all of these reflections.

Note that $\mathcal{W}$ takes weights to weights; in fact, each W_α is an isomorphism of the weight lattice Λ_W. This follows from (10).

Example 6.2. The Weyl group can be relatively simple and familiar — for instance, for $\mathfrak{sl}_n\mathbb{C}$ it is simply S_n, the permutation group on n elements. On the other hand, it can be a little more mysterious. For instance, for $\mathfrak{e}_6$ it is $\mathrm{PSU}_4(2)$—a finite simple group of order 51840.

Highest weights

We would like to introduce the notion of a highest weight, analogously to what worked so well for $\mathfrak{sl}_2\mathbb{C}$, but we face a small conundrum: the weights are no longer just integer but rather points in an integral lattice; how are we to order them?

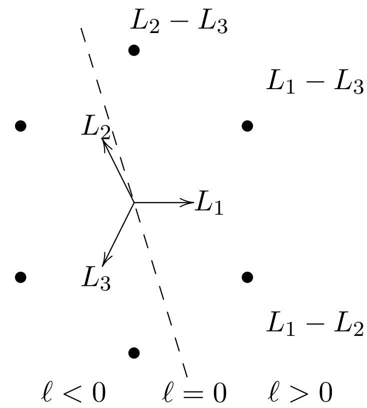
The answer turns out to be very simple: we order them by choosing a height function $\ell : \mathbb{E} \to \mathbb{R}$, a linear function such that $\ell(0) = 0$ and $\ell(\omega) \neq 0$ for any non-zero weight $\omega \in \Lambda_W$.[29] We can then

[28] Recall $H_\alpha = 2T_\alpha/\alpha(T_\alpha)$.

[29] We can always choose such a function precisely because all of the weights lie in an integral lattice in $\mathbb{E}$.

order any collection of weights, say those in $V = \bigoplus_\omega V_\omega$, by $\ell(w)$.

Example 6.3. Here is how we might choose ℓ for $\mathfrak{sl}_3\mathbb{C}$:



So, $L_1 - L_3$ is the highest weight (with respect to ℓ) of the adjoint representation.

Introducing ℓ allows us to do two things: first, we can split the roots as $R = R^+ \cap R^-$ into positive and negative ones. In addition, for any representation $V = \bigoplus_\omega V_\omega$, we can identify a highest weight state as ω_{hw} as the one with maximal ℓ-value.

We can then apply the same logic as in the $\mathfrak{sl}_2\mathbb{C}$ case to obtain the following important result.

Theorem 6.4. *Every irreducible representation has a unique highest weight state, and all other weights are obtained by applying the R^- roots to the highest weight.*

Note that, unlike in the $\mathfrak{sl}_2\mathbb{C}$ case, while the highest weight state is unique, lower weights of an irrep can have non-trivial multiplicity, i.e. $\dim V_\omega > 1$. This leads to an important (but solved!) technical complication in representation theory of a simple $\mathfrak{g}$ versus that of $\mathfrak{sl}_2\mathbb{C}$.

While this is very nice, we have a natural question: how does this structure depend on the choice of $\ell(\omega)$?? The answer is that it does not depend on it, and the key tool in elucidating that is the Weyl group. Before we elaborate on that, we need to introduce another important ℓ-dependent notion.

Definition 6.5. A root $\alpha \in R^+$ is called simple if it cannot be expressed as a linear combination of other positive roots with positive coefficients.

For instance, in our running $\mathfrak{sl}_3\mathbb{C}$ example the roots $L_1 - L_2$ and $L_2 - L_3$ are simple, while $L_1 - L_3$ is not simple.

Theorem 6.6. *The simple roots are linearly independent.*

Proof. The proof follows from properties of roots, and we actually have all the tools to carry it out now; however, we will postpone it until we return for a closer look at roots in the next lecture. □

Since we already know that the roots span $\mathfrak{h}^\vee$, it follows that there must be $\dim \mathfrak{h} = \operatorname{rank} \mathfrak{g}$ simple roots, and they form a basis for $\mathfrak{h}^\vee$. Let's denote the simple roots by α_i, $i = 1, \dots, \operatorname{rank} \mathfrak{g}$. We will not use this definition too much, but it's useful to know.

Definition 6.7. We learned that for any root α we could form $H_\alpha \in \mathfrak{h}$ defined by the property that $\alpha(H) = B(H_\alpha, H)$ for all $H \in \mathfrak{h}$. We will refer to these H_α as the co-roots.

The simple roots, together with the Weyl group will allow us to phrase many questions about roots and weights as properties of the *Weyl chamber*.

Definition 6.8. A Weyl chamber is a subspace of $\mathbb{E} = \Lambda_W \otimes \mathbb{R}$ defined by

$$\mathrm{WC} = \{\omega \in \mathbb{E} \ \ |b(\omega, \alpha_i) \geq 0 \quad \text{for all } i\} \ .$$

In other words, WC consists of all vectors in $\mathbb{E}$ with non-negative inner products with all simple roots. Equivalently, WC is an intersection of rank g half-spaces defined by $\omega \in \mathbb{E}$ satisfying $b(\omega, \alpha_i) \geq 0$. Because the α_i are linearly independent, WC does not contain any half-space and has the form of a *simplicial cone*, i.e. a subspace of $\mathbb{E}$ generated by linearly independent vectors $\omega_i \in \mathbb{E}$ taken with non-negative coefficients. We can now state a fundamental theorem that describes why the choice of the function ℓ is unimportant for many purposes.

Theorem 6.9. *The Weyl group tiles $\mathbb{E}$ with $|WG|$ copies of the Weyl chamber. Given any two weight functions ℓ and ℓ', we have an isomorphism of the corresponding Weyl chambers WC and WC$'$. A weight $\lambda \in WC$ is a highest weight state with respect to ℓ for an irrep V if and only if its Weyl group image λ' is a highest weight state with respect to ℓ'. Thus, all irreps arise from highest weight states in the Weyl chamber, and, moreover, every $\lambda \in \Lambda_W \cap WC$ gives rise to an irrep, which we will denote by Γ_λ.*

We can and will take the ω_i to be an integral basis for $\Lambda_W \cap \mathrm{WC}$ by solving[30]

$$\omega_i(H_{\alpha_j}) = \delta_{ij} \ .$$

These rank g vectors are known as the *fundamental weights*, because all other weights are obtained by taking $\mathbb{Z}$-linear combinations of the ω_i. The highest weight states for all irreps are obtained by taking the coefficients in $\mathbb{Z}_{\geq 0}$. Thus, we can present every irrep in terms of the rank $\mathfrak{g}$ non-negative integers as

$$\Gamma_\omega = \Gamma_{[a_1, a_2, \ldots, a_{\text{rank}\,\mathfrak{g}}]} \ , \tag{11}$$

where $\omega = \sum_i a_i \omega_i$ is the highest weight state. All other weights are obtained by acting on the highest weight with negative simple roots.

Unraveling the definitions, we find that the simple roots are determined as

$$\alpha_j = \sum_k C_{jk} \omega_k \ , \qquad \text{where} \qquad C_{jk} = \frac{2b(\alpha_j, \alpha_k)}{b(\alpha_k, \alpha_k)} \ . \tag{12}$$

The matrix C is known as the *Cartan matrix*. It is non-degenerate, in general not symmetric, and it describes the simple roots in terms of the fundamental weights.

[30] This means that ω_i with integer coefficients generate the full weight lattice Λ_W, while ω_i with non-negative real coefficients generate the full WC.

Definition 6.10. The coefficients of a weight λ written in the basis of the fundamental weights, i.e. the a^i in $\lambda = \sum_i a^i \omega_i$, are known as Dynkin coefficients of the weight.

Exercise 6.11 (Key exercise!!)**.** This is a very important exercise that you should do unless you do not need these lectures. Basically, you should try to run the advertised program for $\mathfrak{sl}_3\mathbb{C}$.

1. Build distinguished $\mathfrak{sl}_2\mathbb{C}$s using the roots and basis we set up above.
2. Compute the Killing form on $\mathfrak{h}$ and b on $\mathfrak{h}^\vee$.
3. What are H_α and T_α?
4. Having ℓ as drawn above, identify the Weyl chamber and check that the Weyl group tiles $\mathbb{E} = \mathbb{R}^2$ with the images of the Weyl chamber.
5. Compute the Cartan matrix.

Solution to exercise

Let's see if we can make sense of our grand program in the simple example of $\mathfrak{sl}_3\mathbb{C}$. Let

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \tag{13}$$

We then have three generators $X_{ij} = E_{ij}$ and three generators $Y_{ij} = E_{ij}^T$.[31] These are definitely roots, since as we observed above, on a generic element H we have $[H, E_{ij}] = (a_i - a_j)E_{ij}$. As expected, we then find that they satisfy $[X_{ij}, Y_{ij}] = H_{ij}$, with

$$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad H_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \qquad H_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \tag{14}$$

and $[H_{ij}, X_{ij}] = +2X_{ij}$, while $[H_{ij}, Y_{ij}] = -2Y_{ij}$. Note that the H_{ij} are not linearly independent: we have $H_{12} + H_{23} = H_{13}$. Clearly these are our three distinguished $\mathfrak{sl}_2\mathbb{C}$s corresponding to the roots $L_1 - L_3$, $L_1 - L_2$ and $L_2 - L_3$.

The Killing form

Let's next work out the Killing form. We could of course do this completely brute-force by writing out explicitly the 8 8×8 matrices $\mathrm{ad}_{X_{12}}, \mathrm{ad}_{X_{23}}, \ldots, \mathrm{ad}_{H_{23}}$. However, we will be a little bit smarter and as a bonus learn a little lesson. Observe that X_{ij}, Y_{ij} and H_{12}, H_{23} form a basis for 3×3 traceless matrices. Concretely, any such M can be uniquely presented as

$$\begin{aligned} M = {} & \mathrm{tr}(MX_{12})Y_{12} + \mathrm{tr}(MX_{13})Y_{13} + \mathrm{tr}(MX_{23})Y_{23} \\ & + \mathrm{tr}(MY_{12})X_{12} + \mathrm{tr}(MY_{13})X_{13} + \mathrm{tr}(MY_{23})X_{23} \\ & + \mathrm{tr}[M(\tfrac{2}{3}H_{12} + \tfrac{1}{3}H_{23})]H_{12} + \mathrm{tr}[M(\tfrac{2}{3}H_{23} + \tfrac{1}{3}H_{12})]H_{23} \ . \end{aligned}$$

[31]We take i, j with $i < j$ for our labels.

So, if we label our 8 generators by $\mathcal{T}^A$

$$\{\mathcal{T}^1, \mathcal{T}^2, \ldots, \mathcal{T}^8\} = \{X_{12}, X_{23}, X_{13}, Y_{12}, Y_{23}, Y_{13}, H_{12}, H_{23}\} \ , \tag{15}$$

then we can uniquely write any M as

$$M = t_A \mathcal{T}^A \ , \qquad t_A = \mathrm{tr}(M\mathcal{T}'_A) \ , \tag{16}$$

where

$$\{\mathcal{T}'_1, \mathcal{T}'_2, \ldots, \mathcal{T}'_8\} = \{Y_{12}, Y_{23}, Y_{13}, X_{12}, X_{23}, X_{13}, \tfrac{2}{3}H_{12} + \tfrac{1}{3}H_{23}, \tfrac{1}{3}H_{12} + \tfrac{2}{3}H_{23}\} \ . \tag{17}$$

The generators satisfy a number of identities.[32] For us, the most relevant are

$$\sum_A \mathcal{T}'_A \mathcal{T}^B \mathcal{T}^A = \sum_A \mathcal{T}^A \mathcal{T}^B \mathcal{T}'_A = -\tfrac{1}{3}\mathcal{T}^B \ , \qquad \sum_A \mathcal{T}'_A \mathcal{T}^A = \sum_A \mathcal{T}^A \mathcal{T}'_A = \tfrac{8}{3}\mathbb{1}_{3\times 3} \ . \tag{18}$$

Armed with this knowledge, let's think about the adjoint action of some $M \in \mathfrak{sl}_3\mathbb{C}$.

$$\mathrm{ad}_M(t_B\mathcal{T}^B) = t_B[M, \mathcal{T}^B] = t_B\,\mathrm{tr}([M, \mathcal{T}^B]\mathcal{T}'_C)\mathcal{T}^C = \left[(\mathrm{ad}_M)^C_B t_C\right]\mathcal{T}^B \ . \tag{19}$$

In other words, when thought of as an 8×8 matrix, (ad_M) has components

$$(\mathrm{ad}_M)^C_B = \mathrm{tr}\left([M, \mathcal{T}^C]\mathcal{T}'_B\right) \ . \tag{20}$$

Hence, if we have $M, N \in \mathfrak{sl}_3\mathbb{C}$, we have

$$B(M,N) = \mathrm{Tr}_{\mathrm{adj}}\,\mathrm{ad}_M\,\mathrm{ad}_N = (\mathrm{ad}_M)^C_B(\mathrm{ad}_N)^B_C = \mathrm{tr}\left([M, \mathcal{T}^C]\mathcal{T}'_B\right)\mathrm{tr}\left([N, \mathcal{T}^B]\mathcal{T}'_C\right) \tag{21}$$

Note that in the final expression the traces are just over the 3×3 matrices! But now we observe

$$\mathrm{tr}\left([M, \mathcal{T}^C]\mathcal{T}'_B\right)\mathcal{T}^B = [M, \mathcal{T}^C] \ , \tag{22}$$

which implies

$$B(M,N) = \mathrm{tr}\left([N, [M, \mathcal{T}^C]]\mathcal{T}'_C\right) \ . \tag{23}$$

Expanding this out and using (18), we obtain

$$B(M,N) = 6\,\mathrm{tr}(MN) \ . \tag{24}$$

So, we have a very simple relation between the Killing form and the trace in the fundamental 3-dimensional representation! As we will see soon, this is no accident, and the result generalizes to any simple LA and any irrep. For instance, for $\mathfrak{sl}_n\mathbb{C}$ we have the relation $B(M,N) = 2n\,\mathrm{tr}_{\boldsymbol{n}}\,MN$, where the trace is taken in the fundamental n-dimensional representation.

For now we can use this to write out the Killing form in great glorious detail. Fixing the order

[32] This might be more familiar in another basis. For instance, using Gell-Mann's Hermitian generators λ^A, which satisfy $\mathrm{tr}\,\lambda^A\lambda^B = 2\delta^{AB}$, there is the identity $\sum_A \lambda^A_{ab}\lambda^A_{cd} = -\frac{1}{3}\sum_A \lambda^A_{ad}\lambda^A_{cb} + \frac{16}{9}\delta_{ad}\delta_{cb}$; more details are in Cahn.

of the generators $\mathcal{T}^A$ as above, we have

$$B(\mathcal{T}^A, \mathcal{T}^B) = 6 \begin{pmatrix} 0 & \mathbb{1}_{3\times3} & 0 & 0 \\ \mathbb{1}_{3\times3} & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{25}$$

In particular, restricting B to $\mathfrak{h}$ with basis H_{12}, H_{23}, we obtain

$$B\big|_{\mathfrak{h}} = 6 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ . \tag{26}$$

The T_α and metric on $\mathbb{E}$

Next, we construct the elements T_α as follows. Recall the definition. For any $\alpha \in \mathfrak{h}^\vee$, we define $T_\alpha \in \mathfrak{h}$ via the relation $B(T_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. In our root basis $L_{1,2,3}$, we write

$$\alpha = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3, \qquad \Longrightarrow \qquad \alpha(H_{ij}) = \alpha_i - \alpha_j \ . \tag{27}$$

Using the Killing form, we then find

$$18 T_\alpha = (\alpha_1 - \alpha_2) H_{12} + (\alpha_2 - \alpha_3) H_{23} + (\alpha_1 - \alpha_3) H_{13} \ . \tag{28}$$

Note that we used $H_{13} = H_{12} + H_{23}$ to write the expression in a slightly more symmetric fashion. We can now easily check that everything works as it's supposed to with the normalization of our distinguished $\mathfrak{sl}_2\mathbb{C}$s. In particular, if we use the shorthand $L_{ij} = L_i - L_j$ for the roots, then $[X_{ij}, Y_{ij}] = B(X_{ij}, Y_{ij}) T_{L_{ij}}$ is consistent with

$$T_{L_{ij}} = \tfrac{1}{6} H_{ij} \ , \qquad B(X_{ij}, Y_{ij}) = 6 \ . \tag{29}$$

The metric b on $\mathbb{E}$ takes a very simple explicit form:

$$b(\alpha, \beta) = \alpha(T_\beta) = \tfrac{1}{18} \left[(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) + (\alpha_2 - \alpha_3)(\beta_2 - \beta_3) + (\alpha_1 - \alpha_3)(\beta_1 - \beta_3) \right] \ . \tag{30}$$

This finally makes it clear why we like to draw our picture of $\mathbb{E}$ as in Example 6.3. If we take $L_1 = (\frac{1}{3}, 0)$, and L_2 and L_3 spaced by $2\pi/3$ rotations from it, the metric $b(\alpha, \beta)$ is just the Euclidean metric on the plane!

The weight lattice, fundamental weights, and Cartan matrix

The weight lattice is the set of all vectors that have integer pairing with the H_{ij}. Clearly these are given by $\omega_i L_i$, with $\omega_i \in \mathbb{Z}$. With ℓ chosen as in Example 6.3, the simple roots are L_{12} and L_{23}, and the corresponding *co-roots* are H_{12} and H_{23}. The fundamental weights ω_{ij} are then defined by

$$\omega_{12}(H_{12}) = 1 \ , \qquad \omega_{12}(H_{23}) = 0 \ , \qquad \omega_{23}(H_{12}) = 0 \ , \qquad \omega_{23}(H_{23}) = 1 \ . \tag{31}$$

The solution to this is $\omega_{12} = L_1$ and $\omega_{23} = L_1 + L_2$. We can invert the relation to express the simple roots in terms of the fundamental weights:

$$L_{12} = 2\omega_{12} - \omega_{23} \ , \qquad L_{23} = -\omega_{12} + 2\omega_{23} \ , \tag{32}$$

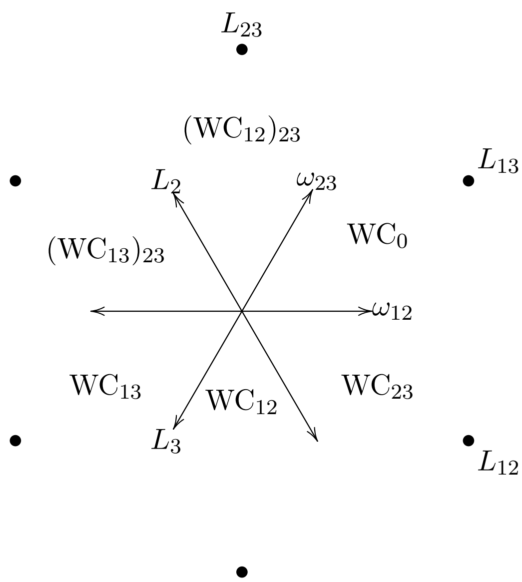
from which we extract the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ . \tag{33}$$

Note that in this case C turns out to be symmetric.[33]

Weyl chamber and reflections

Finally, we take a look at the Weyl chamber and its images under the Weyl group. This is pretty easy since we are reflecting with respect to a nice Euclidean metric, and all we need to do is keep track of where the generating vectors of the WC, ω_{12} and ω_{23} are sent by reflections.



[33]Note that simple $\mathfrak{g}$ with a symmetric Cartan matrix are known as *simply-laced*. The simply-laced $\mathfrak{g}$ are $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}(2n)$, and $\mathfrak{e}_{6,7,8}$.

7 Root systems and Dynkin diagrams

In this lecture we describe the complete classification of simple complex LAs in terms of root systems. We begin with a few more comments about the properties of the Weyl group, then discuss root systems and their classification.

7.1 A little more on the Weyl group

Recall that we learned that once we fixe a Cartan subalgebra of a simple LA $\mathfrak{g}$, $\mathbb{E} = \Lambda_W \otimes \mathbb{R} \simeq \mathbb{R}^{\text{rank}\,\mathfrak{g}}$ is endowed with a positive-definite metric $b(\alpha, \beta)$. A key part in the discussion was played by the distinguished $\mathfrak{sl}_2\mathbb{C}$s, one for every root α. For any such $\mathfrak{s}_\alpha$ we defined the diagonal generator $H_\alpha \subset \mathfrak{s}_\alpha$ and showed that for any $\beta \in \mathbb{E}$ $\beta(H_\alpha) = 2b(\beta, \alpha)/b(\alpha, \alpha)$. Representation theory of $\mathfrak{sl}_2\mathbb{C}$ then guarantees that for any weight $\omega \in \Lambda_W$ $\omega(H_\alpha) \in \mathbb{Z}$ — a fact we used repeatedly above.

However, we know another fact about $\mathfrak{sl}_2\mathbb{C}$ representations: for any representation the string of $\mathfrak{sl}_2\mathbb{C}$ weights is an uninterrupted string of integers symmetric about the origin, i.e. ω belongs to an uninterrupted string

$$\omega + p\alpha \ , \quad \omega + (p-1)\alpha \ , \quad \ldots \ , \quad \omega - (q-1)\alpha \ , \quad \omega - q\alpha$$

for two non-negative integers p, q. The corresponding eigenvalues of $H_\alpha \in \mathfrak{s}_\alpha$ are then (using $\alpha(H_\alpha) = 2$)

$$\omega(H_\alpha) + 2p \ , \quad \omega(H_\alpha) + 2(p-1) \ , \quad \ldots \ , \quad \omega(H_\alpha) - 2(q-1) \ , \quad \omega(H_\alpha) - 2q \ ,$$

and by $\mathfrak{sl}_2\mathbb{C}$ properties it is symmetric about its origin, so that

$$q = p + \omega(H_\alpha) \ .$$

This means the string of weights is then

$$\omega + p\alpha \ , \quad \omega + (p-1)\alpha \ , \quad \ldots \ , \quad \omega - \omega(H_\alpha)\alpha - (p-1)\alpha \ , \quad \omega - \omega(H_\alpha)\alpha - p\alpha \ .$$

But now consider the Weyl group generator W_α, which acts as a reflection

$$W_\alpha(\omega) = \omega - \frac{2b(\omega, \alpha)}{b(\alpha, \alpha)}\alpha = \omega - \omega(H_\alpha)\alpha \ .$$

Evidently,

1. the Weyl generator W_α reflects the string of weights $w + p\alpha, \ldots, w - q\alpha$;
2. if ω is a weight of a representation V then so are all of its Weyl group images;
3. in particular, if α, β are roots, then $W_\alpha(\beta)$ is also a root.

Root systems

Let R be a set of roots with respect to a choice of some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We showed that the roots satisfy the following four properties:

1. R spans $\mathfrak{h}^\vee$, i.e. $\mathbb{E} = \Lambda_R \otimes \mathbb{R}$ has dim = rank $\mathfrak{g}$;
2. if $\alpha \in R$ then $k\alpha \in R$ if and only if $k = \pm 1$;
3. W_α maps R to itself for every $\alpha \in R$;
4. $n_{\beta\alpha} = \frac{2b(\beta,\alpha)}{b(\alpha,\alpha)} \in \mathbb{Z}$.

Definition 7.1. Fix a Euclidean space $\mathbb{E}$ with a positive-definite metric b. Then an abstract root system is any set of vectors $R \subset \mathbb{E}$ satisfying the four properties above.

We then have a remarkable theorem

Theorem 7.2 (Dynkin)**.** *Up to isomorphism all abstract root systems are classified and are in 1:1 correspondence with complex simple LAs.*

In the rest of the lecture we will sketch the classification method and state the result.

Angles between roots

We begin by using the properties of the metric. Given any two roots α, β, the angle θ between them is constrained by $n_{\beta\alpha} = 2\cos\theta\frac{|\beta|}{|\alpha|}$. Integrality of $n_{\beta\alpha}$ severely constrains possible values of θ, since

$$n_{\beta\alpha}n_{\alpha\beta} = 4\cos^2\theta \in \{0, 1, 2, 3, 4\} \ .$$

The last possibility is uninteresting, since it only realized by $\alpha = \pm\beta$. For the remaining 4 possibilities we can make a little table

$n_{\beta\alpha}n_{\alpha\beta}$	$\lvert\beta\rvert/\lvert\alpha\rvert$	θ
0	anything	$\pi/2$
1	1	$2\pi/3$ or $\pi/3$
2	$\sqrt{2}$	$3\pi/4$ or $\pi/4$
3	$\sqrt{3}$	$5\pi/6$ or $\pi/6$

More properties of roots

To proceed, choose $\ell : \Lambda_R \to \mathbb{R}$ to split the roots into positive and negative: $R = R^+ \cup R^-$ and define simple roots $\subset R^+$ such that all positive roots are positive combinations of simple roots. We then make a string of observations.

5. If $\beta \neq \pm\alpha$ then $\beta + p\alpha, \ldots, \beta - q\alpha$ string has at most 4 elements, and $q - p = n_{\beta\alpha}$.

6. It follows that if $\beta \neq \pm\alpha$ are roots then

$$b(\beta,\alpha) > 0 \implies \alpha - \beta \in R$$
$$b(\beta,\alpha) < 0 \implies \alpha + \beta \in R$$
$$b(\beta,\alpha) = 0 \implies (\alpha \pm \beta) \in R \quad \text{or} \quad (\alpha \pm \beta) \notin R\ .$$

7. But, on the other hand, if $\alpha \neq \beta$ are simple roots then $\alpha - \beta \notin R$ since $\alpha - \beta$ is neither positive (because $\alpha = \beta + (\alpha - \beta)$) or negative (because $\beta = \alpha + (\beta - \alpha)$.)

8. Hence, for simple roots $\theta \geq \pi/2$.

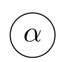
9. By construction simple roots are all vectors lying above the hyperplane $\ell = 0$, and we just showed they have pairwise $\theta \geq \pi/2$. This implies that the simple roots are linearly independent.

10. So, finally, we conclude that there are $\dim \mathbb{E}$ simple roots. Each positive root is written uniquely as a non-negative integral combination of simple roots, and no root has mixed signs when written in terms of the simple roots.

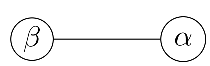
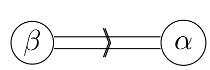
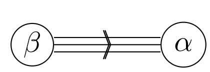
Exercise 7.3. Show that the assumptions of observation 9 imply its conclusion.

Dynkin diagrams

Dynkin came up with a beautiful graphical way to encode the properties of the simple root arrangements. Each simple root is labeled by a circle in the plane; if we need to specify a label for the root, we just place it in the circle like so:



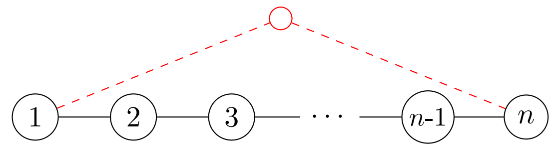
Having drawn a circle for every simple root in the root system, the pairwise angles are indicated by connecting them with lines according to the following scheme:

$n_{\beta\alpha} = n_{\alpha\beta} = 0$		$\theta = \pi/2$	
$n_{\beta\alpha} = n_{\alpha\beta} = 1$		$\theta = 2\pi/3$	$\lvert\beta\rvert = \lvert\alpha\rvert$
$n_{\beta\alpha} = -2\ , n_{\alpha\beta} = -1$		$\theta = 3\pi/4$	$\lvert\beta\rvert = \sqrt{2}\lvert\alpha\rvert$
$n_{\beta\alpha} = -3\ , n_{\alpha\beta} = -1$		$\theta = 5\pi/6$	$\lvert\beta\rvert = \sqrt{3}\lvert\alpha\rvert$

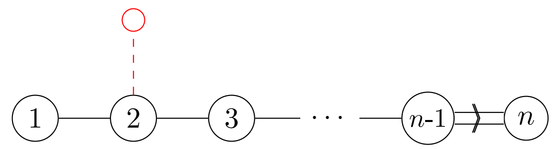
Note that in each case the arrow points to the shorter root.

With this notation we can now present Dynkin's classification. The abstract root systems consist of four infinite families, and 5 exceptions. We write down the Dynkin diagrams for each of the allowed root systems. For future reference we include more information indicated with the dashed lines and colored nodes. For now the reader should focus on the solid lines and black nodes.

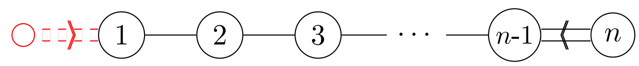
A_n $\quad n \geq 1$ $\quad \mathfrak{sl}_{n+1}\mathbb{C}$



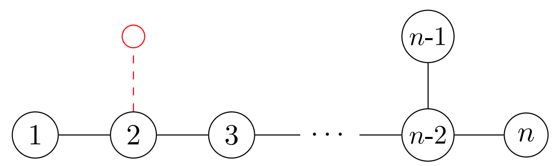
B_n $\quad n \geq 2$ $\quad \mathfrak{so}_{2n+1}\mathbb{C}$



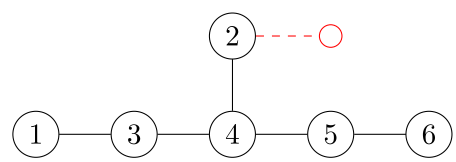
C_n $\quad n \geq 2$ $\quad \mathfrak{sp}_{2n}\mathbb{C}$



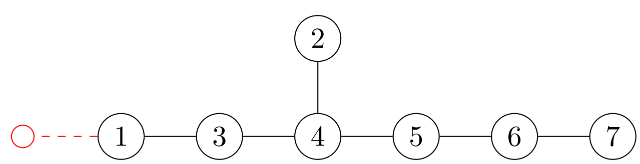
D_n $\quad n \geq 3$ $\quad \mathfrak{so}_{2n}\mathbb{C}$



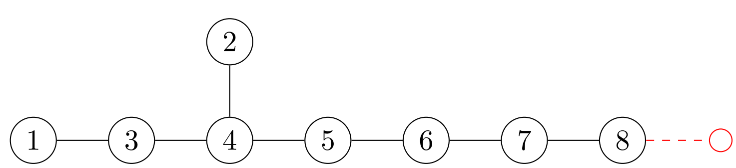
E_6 $\quad \mathfrak{e}_6$



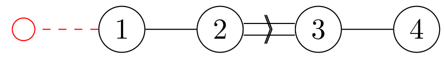
E_7 $\quad \mathfrak{e}_7$

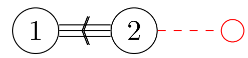


E_8 $\quad \mathfrak{e}_8$



Finally, we also have

F_4 $\mathfrak{f}_4$ 

G_2 $\mathfrak{g}_2$ 

And those are all the possible root systems! We will not have time to go through the proof of this classification, but actually it is rather simple given the facts above the root systems that we gave above. Some of the details are sketched in Chapter IX of Cahn.

Comments on diagrams

1. Every one of these root systems is actually realized by a LA.

2. If one accepts on faith the identifications of root systems and LAs as indicated in the table, then a number of familiar isomorphisms become readily apparent:[34]

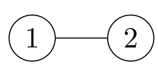
$$\mathfrak{so}(3) \simeq \mathfrak{sl}_2\mathbb{C} \simeq \mathfrak{sp}(1) \qquad \mathfrak{so}(5) \simeq \mathfrak{sp}_4\,\mathbb{C} = \mathfrak{sp}(2) \qquad \mathfrak{so}(6) \simeq \mathfrak{sl}_4\mathbb{C} \qquad \mathfrak{so}(4) \simeq \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}\ .$$

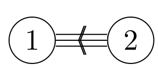
3. The diagrams above use a particular numbering scheme for the simple roots: there are a number of these on the market; we are following the Bourbaki labeling scheme.

4. The diagrams without double or triple lines are known as "simply laced," as are the corresponding ADE LAs. Many interesting mathematical structures turn out to have an ADE classification. For instance, finite subgroups of SU(2) are in $1:1$ correspondence with the ADE LAs. More exotic and more physical objects include the ADE classfication of $N=2$ unitary superconformal minimal models in $d=2$.

5. The diagrams with the "extra" red node are known as "affine" or "extended" Dynkin diagrams. They have many applications: we will see how they help to describe sub-algebras of LAs.

6. Each diagram encodes the Cartan matrix (and vice versa!). So, for example, we have


$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$


$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Exercise 7.4. Using the Dynkin diagrams derive the Cartan matrices for the remaining exceptional LAs $\mathfrak{e}_{6,7,8}$ and $\mathfrak{f}_4$.

[34] In hopes of lessening confusion, we wrote two different notations for the $\mathfrak{sp}$ LA, i.e. $\mathfrak{sp}_{2n}\,\mathbb{C}$ vs $\mathfrak{sp}(n)$. These really are the same as complex LAs— the defining representation has complex dimension $2n$, and so on.

8 Basic properties of irreducible representations

In this lecture we use the language of roots and weights to answer some simple questions about irreducible representations of a simple complex LA $\mathfrak{g}$.

Weights from the highest weight

Let's begin by reviewing the structure described in theorem 6.9 . Given $\mathfrak{g}$ and a choice of simple roots α_i, $i = 1, \ldots, \operatorname{rank}\mathfrak{g}$, we construct the Weyl chamber WC — a pointed rational cone in $\mathbb{E} = \Lambda_W \otimes \mathbb{R}$. $\Lambda_W \cap \mathrm{WC}$ is then generated by the fundamental weights ω_i with non-negative integral coefficients, i.e. every $\lambda \in \Lambda_W \cap \mathrm{WC}$ is of the form $\lambda = \sum_i a^i \omega_i$, where $a^i \in \mathbb{Z}_{\geq 0}$. We have three more key properties of the fundamental weights:

1. the Cartan matrix describes the simple roots in terms of the fundamental weights: $\alpha_i = \sum_j C_{ij}\omega_j$;
2. $\omega_i(H_{\alpha_j}) = \delta_i^j$, which implies $\lambda(H_{\alpha_j}) = a^j$;
3. irreps of $\mathfrak{g}$ are in 1:1 correspondence with $\lambda \in \Lambda_W \cap \mathrm{WC}$, where λ is the highest weight state of

$$\Gamma_\lambda = V_\lambda \oplus V_{\mu_1} \oplus V_{\mu_2} \oplus \cdots \oplus V_{\mu_s} \ .$$

We face two basic questions. First, how do we determine the remaining lower weights $\mu_1, \ldots, \mu_s$? Second, while the highest weight vector is unique, i.e. $\dim V_\lambda = 1$, in general (this is a difference from $\mathfrak{sl}_2\mathbb{C}$!) $\dim V_\mu$ can be bigger than one; so, how can we determine this dimension? We will now tackle these questions.

To determine the remaining weights in Γ_λ, we follow the same course as we did for $\mathfrak{sl}_2\mathbb{C}$: we simply need to repeatedly apply the lowering operators Y_α to V_λ to obtain $V_{\lambda-\alpha}$, and all we need to do is just figure out which of these vector spaces are non-empty.[35] Clearly it is sufficient to restrict attention to the simple roots, since all others are just linear combinations of the simple roots. We then use the basic tool that we already encountered in the previous lecture: a weight $\lambda = \sum_i a^i\omega_i$ belongs to an uninterrupted string of the form

$$\lambda + p\alpha_j, \lambda + (p-1)\alpha_j, \ldots, \lambda - (q+1)\alpha_j, \lambda - q\alpha_j$$

if and only if $q - p = \lambda(H_{\alpha_j}) = a^j$. Hence, if λ is a highest weight state, then $p = 0$ and α_j can be subtracted a^j times; doing it one more time leads to annihilation (of the corresponding vector space, nothing worse). So, our weights in Γ_λ will be of the form $\lambda - \sum_j m^j\alpha_j$.

Definition 8.1. The level of a weight $\lambda - \sum_j m^j\alpha_j$ is $\sum_j m^j$.

We then have a simple algorithm to obtain all the weights in Γ_λ. At level 0 we simply have λ. If at level n we have μ with some $a^j > 0$, then we obtain weights $\mu - k\alpha_j$ for $k = 1, \ldots, a^j$ at levels

[35] The highest weight space is of course annihilated by all the raising operators.

$n+1, n+2, \ldots, n+k$. We simply write down all of these weights and proceed a level down until it is not possible to subtract any more roots.[36]

Example 8.2. Let's get a little practice with $\mathfrak{g} = \mathfrak{sl}_3\mathbb{C}$. Here

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (C) \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Now let's write down some lists of weights!

$\Gamma_{[10]}$		
0	[1 0]	λ
1	[−1 1]	$\lambda - \alpha_1$
2	[0 −1]	$\lambda - \alpha_1 - \alpha_2$

$\Gamma_{[01]}$		
0	[0 1]	λ
1	[1 −1]	$\lambda - \alpha_2$
2	[−1 0]	$\lambda - \alpha_2 - \alpha_1$

These were pretty easy, because we never had to worry about a string of length more than one, and there is no possibility of multiplicity: since there is just one way to get to every weight, $\dim V_\mu = 1$ for every weight here. So, in fact, these are the familiar fundamental and anti-fundamental representations of $\mathfrak{sl}_3\mathbb{C}$. Using the usual physics labeling by dimension, we have $\Gamma_{[10]} = \mathbf{3}$ and $\Gamma_{[01]} = \overline{\mathbf{3}}$.

For a more sophisticated example, we consider $\Gamma_{[20]}$. Since $a^1 = 2$ here, we can subtract α_1 twice, so our first step is to write down

$\Gamma_{[20]}$	first step	
0	[2 0]	λ
1	[0 1]	$\lambda - \alpha_1$
2	[−2 2]	$\lambda - 2\alpha_1$

Now we come to level 1, and we see that we can subtract α_2 once from the unique level 1 weight to obtain

$\Gamma_{[20]}$	first step	
0	[2 0]	λ
1	[0 1]	$\lambda - \alpha_1$
2	[−2 2][1 −1]	$\lambda - 2\alpha_1, \lambda - \alpha_1 - \alpha_2$

Next, we come to level 2. We see that we can either subtract α_2 twice from [−2 2] or α_1 once from [1 −1]. In fact, this is all we can do, so that we are done! However, we observe that there are two ways to get to the weight [−1 0], so that there is a possible multiplicity (of at most 2) for the weight at level 3. In this case the multiplicity turns out to be 1, so that the full weight table is

[36] By writing down the full string of weights for every $a^j > 0$, i.e. going "all the way down" every time, we avoid having to keep track of a non-trivial p. If this sounds cryptic, hopefully examples will make it clear.

simply

$\Gamma_{[20]}$		
0	$[\,2\ \ 0]$	λ
1	$[\,0\ 1]$	$\lambda-\alpha_1$
2	$[-2\ 2][\,1\ -1]$	$\lambda-2\alpha_1, \lambda-\alpha_1-\alpha_2$
3	$[-1\ 0]$	$\lambda-2\alpha_1-\alpha_2$
4	$[\,0\ -2]$	$\lambda-2\alpha_1-2\alpha_2$

This is another familiar representation: $\Gamma_{[20]} = \mathbf{6} = \mathrm{Sym}^2\,\mathbf{3}$. As our final example, we work give the weights for $\Gamma_{[11]}$:

$\Gamma_{[11]}$	
0	$[\,1\ \ 1]$
1	$[-1\ 2][\,2\ -1]$
2	$[\,0\ 0]^{\oplus 2}$
3	$[-2\ 1][\,1\ -2]$
4	$[\,-1\ -1]$

In this case the weight [00] has multiplicity 2, as indicated by the superscript.

Exercise 8.3. Work out the weights for $\Gamma_{[10]}$ and $\Gamma_{[01]}$ for $\mathfrak{g}_2$.

The Freudenthal recursion formula

There are a number of tricks for determining the multiplicity of the weights. For instance, the weight diagram has to be "spindle-shaped,", i.e. the number of weights at level $\ell+1$ has to be greater than or equal to that at level ℓ for $\ell < \ell_{\max}/2$, and the number of weights at level ℓ is the same as that at $\ell_{\max}-\ell$. These tricks are often sufficient to determine all of the multiplicities, but for large irreps they are not sufficient. In that case a recursive formula, determines the multiplicities. Let N_μ be the multiplicity of weight μ in Γ_λ. Then

$$N_\mu\left[b(\lambda+\delta,\lambda+\delta)-b(\mu+\delta,\mu+\delta)\right] = 2\sum_{\alpha\in R^+,k>0} N_{\mu+k\alpha}b(\mu+k\alpha,\alpha)\ ,$$

where δ is the Weyl vector to be defined below. The point is that since the right-hand-side involves weights of strictly lower level than μ, and we know $N_\lambda = 1$, we can solve it recursively to obtain every multiplicity. In practice, once you are dealing with irreps where you really need this, you may as well use one of the excellent software packages that will carry this out without algebra mistakes.

Roots from simple roots

While we have described the weights of any irreps and gave a formula for their multiplicity, we have yet to determine what the reader may well consider to be a more fundamental structure: the roots of $\mathfrak{g}$! Fortunately, this can also be done algorithmically. In a sense, we have already done so, since all we need to know is the highest weight of the adjoint representation, but we can also work out the roots more directly. A few facts help our goal to determine all $\beta \in R^+$:

1. once we know the positive roots $\beta \in R^+$ we also know the negative roots;
2. every β can be written as $\beta = \sum_j k_j \alpha_j$ with $k_j \in \mathbb{Z}_{\geq 0}$;
3. $\alpha_i - \alpha_j$ is not a root;
4. $2\alpha_i$ is not a root.

As in the case of weights, we have a notion of a level: $\beta = \sum_j k_j \alpha_j$ is said to have level $\sum_j k_j$. We then follow a very similar procedure as for the weights, working level by level. At level 1 we just have the simple roots: $\{\alpha_1, \ldots, \alpha_{\text{rank}\,\mathfrak{g}}\}$. Suppose we know the roots up to level n. We need to decide whether $\beta + \alpha_j$ is a root, and to do so we again use the uninterrupted string. Since we know everything up to level n, we know all the negative entries in

$$\beta - q\alpha_j, \beta - (q-1)\alpha_j, \ldots, \beta, \ldots, \beta + p\alpha_j \ .$$

Hence, we can use $q - p = \beta(H_{\alpha_j})$ to determine the possible roots at level $n+1$. As in the case of weights, we can avoid having to think about non-zero q if we always extend the string as far up as possible.

Example 8.4. For instance, we have

$\mathfrak{sl}_3\mathbb{C}$	
1	[2 −1] [−1 2]
2	[1 1]

$\mathfrak{sp}(4)$	
1	[2 −1] [−2 2]
2	[0 1]
3	[2 0]

$\mathfrak{so}(8)$	
1	[2 −1 0 0] [−1 2 −1 −1] [0 −1 2 0] [0 −1 0 2]
2	[1 1 −1 −1] [−1 1 1 −1] [−1 1 −1 1]
3	[1 0 1 −1] [1 0 −1 1] [−1 0 1 1]
4	[1 −1 1 1]
5	[0 1 0 0]

Exercise 8.5. Work out the roots of $\mathfrak{g}_2$. Compare them to the weights of $\Gamma_{[01]}$.

The Weyl dimension formula

Having worked out the weights and roots, we now describe a remarkably simple formula for the dimension of an irrep Γ_λ. We begin with a definition.

Definition 8.6. The Weyl vector δ is half the sum of the positive roots: $\delta = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$.

This is a rather remarkable object. We have already seen that it makes an appearance in the Freudenthal multiplicity formula quoted above, and it will show up in a number of other useful formulas. Even aside from this, we have the following non-trivial result:

Theorem 8.7. *The Weyl vector has Dynkin coefficients* $[1, 1, \ldots, 1]$, *i.e.* $\delta = \sum_i \omega_i$.

The proof is neat and illustrates the utility of the Weyl group. It is based on a little lemma:

Lemma 8.8. *Let β be a positive root and α_i a simple root with $\beta \neq \alpha_i$. Then the Weyl reflection $W_{\alpha_i}(\beta)$ is another positive root.*

To prove this, we recall that a Weyl reflection of a root is also a root, and for $\beta = \sum_j k_j \alpha_j$, with $k_j \geq 0$, we have

$$W_{\alpha_i}(\beta) = \beta - \frac{b(\beta, \alpha_i)}{b(\alpha_i, \alpha_i)} \alpha_i = \sum_{j \neq i} k_j \alpha_j + c\alpha_i$$

for some constant c. But, a root, when written in terms of simple roots has either all positive or all negative coefficients. Since some $k_j > 0$ by assumption, $W_{\alpha_i}(\beta)$ must be a positive root.

Proof. Now we can prove our theorem about δ. The trick is to compute $b(W_{\alpha_i}(\delta), \alpha_i)$ in two different ways. First, because the metric b is preserved by reflections, we have

$$b(W_{\alpha_i}(\delta), \alpha_i) = b(\delta, W_{\alpha_i}(\alpha_i)) = b(\delta, -\alpha_i) = -b(\delta, \alpha_i) \ .$$

Second, using our little lemma we see that $W_{\alpha_i}(\delta) = \delta - \alpha_i$, so that

$$b(W_{\alpha_i}(\delta), \alpha_i) = b(\delta - \alpha_i, \alpha_i) = b(\delta, \alpha_i) - b(\alpha_i, \alpha_i) \ .$$

Putting the two results together, we see that the i-th Dynkin coefficient of δ is given by

$$\frac{2b(\delta, \alpha_i)}{b(\alpha_i, \alpha_i)} = 1 \ .$$

□

Armed with a good understanding of the wonderful Weyl vector, we can now state Weyl's dimension formula.

Theorem 8.9. *If Γ_λ is an irrep with highest weight λ, then*

$$\dim \Gamma_\lambda = \prod_{\alpha \in R^+} \frac{b(\alpha, \lambda + \delta)}{b(\alpha, \delta)} \ .$$

The proof of this is quite ingenious and relatively easy to follow — take a look at Cahn's book for details. The basic idea is to study characters of $\Gamma_\lambda = \bigoplus_\mu V_\mu$, i.e. for $\rho \in \mathfrak{h}^\vee$, we write down

$$\chi_{\Gamma_\lambda}(\rho) = \sum_\mu \dim V_\mu e^{b(\mu, \rho)} \ .$$

It is not too hard to prove that χ is invariant under the action of the Weyl group, and by decomposing the sum into Weyl group orbits, one can obtain a closed form for this object. Finally, by carefully taking the $\rho \to 0$ limit, one obtains

$$\lim_{\rho \to 0} \chi_{\Gamma_\lambda}(\rho) = \sum_\mu \dim V_\mu = \dim \Gamma_\lambda \ .$$

Exercise 8.10. Apply the general formula to the case of $\mathfrak{sl}_3\mathbb{C}$ to discover

$$\dim \Gamma_{[m_1 \ m_2]} = \tfrac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2) \ .$$

Some tensor products

Once we know the weights and multiplicities of various irreps, it is easy to obtain new representations by taking tensor products.[37] The basic idea is very easy if

$$\Gamma_\lambda = \bigoplus_\mu V_\mu \ , \qquad \Gamma_{\lambda'} = \bigoplus_\nu V'_\nu \ ,$$

then

$$\Gamma_\lambda \otimes \Gamma_{\lambda'} = \bigoplus_{\mu,\nu} V_\mu \otimes V'_\nu = \bigoplus_{\mu,\nu} W_{\mu+\nu} \ ,$$

where $\dim W_{\mu+\nu} = \dim V_\mu + \dim V'_\nu$. Similarly, we have (with some slightly severe abuse of notation)

$$\mathrm{Sym}^2 \Gamma_\lambda = \bigoplus_{\mu \geq \nu} V_\mu \otimes V_\nu \ , \qquad \wedge^2 \Gamma_\lambda = \bigoplus_{\mu > \nu} V_\mu \otimes V_\nu \ . \tag{34}$$

A note on the abuse: really we mean the following: label the weights $\mu_1, \ldots, \mu_s$; then take the sum over $V_{\mu_m} \otimes V_{\mu_n}$ with $m \geq n$ for Sym and $m > n$ for $\wedge$.

Example 8.11. Using the weight tables we derived above for $\mathfrak{sl}_3\mathbb{C}$, we can give some familiar examples. In each case we list the weights with multiplicities.

$$\Gamma_{[10]} \otimes \Gamma_{[01]} : \begin{array}{c} [1\ 1] \\ [2\ -1]\ [-1\ 2] \\ [0\ 0]^{\oplus 3} \\ [-2\ 1]\ [1 -2] \\ [-1 -1] \end{array} = \Gamma_{[11]} \oplus \Gamma_{[00]} \ .$$

This is of course the familiar $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

As another example, we can decompose $\Gamma_{[10]} \otimes \Gamma_{[10]} = \mathrm{Sym}^2 \Gamma_{[10]} \oplus \wedge^2 \Gamma_{[10]}$:

$$\mathrm{Sym}^2 \Gamma_{[10]} : \begin{array}{c} [2\ 0] \\ [0\ 1]\ [-2\ 2] \\ [-1\ -1]\ [-1\ 0] \\ [0\ -2] \end{array} = \Gamma_{[20]} \ , \qquad \wedge^2 \Gamma_{[10]} : [0\ 1]\ [1\ -1]\ [-1\ 0] = \Gamma_{[01]} \ .$$

In this case we learn that $\mathrm{Sym}^2\, \mathbf{3} = \mathbf{6}$ and $\wedge^2 \mathbf{3} = \overline{\mathbf{3}}$.

[37] The reader may want to take a look back at Representation tricks I, as well as review the $\mathfrak{sl}_2\mathbb{C}$ story from above.

9 Representation tricks II

In this lecture we introduce a few more notions useful in classifying and distinguishing representations. We focus on four tools: Schur's lemma, unitary representations, bilinear invariant tensors, and the index of representation.

Schur's lemma

We begin with an incredibly useful result that one encounters again and again in representation theory.

Theorem 9.1 (Schur's lemma)**.** *Let V and W be irreps of a group G, and let $\varphi : V \to W$ be a G-linear map. Then φ is either an isomorphism or 0, and in the former case $\varphi = \lambda \mathbb{1}$.*

Proof. G-linearity of φ means $\rho_W(x)\varphi = \varphi\rho_V(x)$ for all $x \in G$. This implies that $\ker\varphi$ and $\operatorname{im}\varphi$ are representations of G, and since V and W are irreps by assumption, the only possibilities are that φ is an isomorphism or $\varphi = 0$. Suppose the former and let $\lambda \in \mathbb{C}^*$ be an eigenvalue of φ, but then $\ker(\varphi - \lambda\mathbb{1}) \neq 0$ is a representation of G, and irreducibility requires it to be all of V, i.e. $\varphi = \lambda\mathbb{1}$. □

Unitarity of representations of a compact Lie group

Let G be a compact LG with a simple LA $\mathfrak{g}$. While $\mathfrak{g}$ is necessarily a real LA, the representations V are in general complex, i.e. the matrices $\rho_V(x)$ are in general complex, and it does not make sense to have them act on a real vector space. Complex vector spaces admit Hermitian metrics. Can we have a sensible metric for V as a representation of G? We make an optimistic definition:

Definition 9.2. A representation V of G is unitary if it admits a G-invariant Hermitian metric H, i.e. for any $v, w \in V$ $H(v, w)$ linear in w and anti-linear in v, with $H(v, v) \geq 0$ with equality iff $v = 0$, satisfying $H(v, w) = H(X \cdot v, X \cdot w)$ for all $X \in G$.[38]

We now state an important theorem, used to great effect by Weyl (and hence intimately related to "Weyl's unitarity trick.").

Theorem 9.3. *Every finite-dimensional representation of a compact LG is unitary.*

Proof. (a sketch) There exists a G-invariant integration measure on G, known as the Haar measure $\mu(X)$. Hence, if we pick any Hermitian metric H_0 on V, then

$$H(v, w) = \int_G \mu(Y)\ H_0(Y \cdot v, Y \cdot w)$$

is G-invariant, since

$$H(X \cdot v, Y \cdot w) = \int_G \mu(Y)\ H_0(YX \cdot v, YX \cdot w) = \int_G \mu(Z)\ H_0(Z \cdot v, Z \cdot w)\ ,$$

where in the last equality we changed integration variables to $Z = YX$ and used invariance of the measure. Since G is compact, the integral exists and H is well-defined. □

[38] Note we are abusing notation a little bit: we are writing $\rho_V(X)v$ as $X \cdot v$.

Once we have this invariant Hermitian metric, we can without loss of generality choose a basis on V such that $H = \mathbb{1}$, so that $H(v, w) = v^\dagger w$. This is a very convenient choice, since with it, the invariance implies that $\rho_V(X)^\dagger \rho_V(X) = \mathbb{1}$ for all $X \in G$. In other words, $\rho_V(X) \in \mathrm{U}(V) \subset \mathrm{GL}(V)$. Linearizing this around the identity in G, we have $\rho_V(X) = e^{\theta\mathcal{T}} = \mathbb{1} + \theta\mathcal{T}$, where $\mathcal{T}$ is a generator of the LA $\mathfrak{g}$. Invariance of the inner product now reads

$$\delta_\theta(v^\dagger w) = \delta_\theta v^\dagger w + v^\dagger \delta_\theta w = \theta v^\dagger(\mathcal{T}^\dagger + \mathcal{T})w = 0 \ .$$

In other words, we can take the the LA generators to be be Hermitian.[39] Moreover, if $\mathfrak{g}$ is simple, then $\mathcal{T}$ is also traceless, i.e. $\mathcal{T} \in \mathfrak{su}(V)$.

Conjugate and dual representations

Given a representation V with transformations $\delta_\theta v = \theta\mathcal{T}v$, we can define the conjugate representation $\overline{V}$ by the transformations $\delta_\theta \overline{v} = \theta\mathcal{T}^*\overline{v}$. It is perhaps even more natural to define the dual representation $V^\vee$ by demanding that the linear action of $w \in V^\vee$ on V is $\mathfrak{g}$-invariant. This fixes the transformation as $\delta_\theta w = -\,{}^t\mathcal{T}w$. But, since $\mathcal{T}$ is anti-Hermitian, we see that the conjugate and dual representations are in fact isomorphic. We could say this more invariantly by noting that the Hermitian metric yields an isomorphism $V \to \overline{V}^\vee$.

Once we know the action of $\mathfrak{g}$ on V, we know it on the induced representations $V^\vee$ and $\overline{V}$. In fact, we also know it on all tensor products of V and $V^\vee$.[40] Let's consider a particularly simple plethysm $W = V \otimes V^\vee$. If $\dim V = n$, we can think of W as $n \times n$ matrices or be a bit more sophisticated and write $W = \mathrm{End}(V)$. Either way, the induced action has the following form. If $M \in W$ then

$$\delta_\theta M = \theta[\mathcal{T}, M] \ .$$

This immediately shows that if $\dim V > 1$, then since M includes all the generators $\mathcal{T}$ and the identity, we see

$$V \otimes V^\vee = V \otimes \overline{V} \supset \mathrm{adj}(\mathfrak{g}) \oplus \mathbf{1} \ ,$$

i.e. the decomposition into irreducible representations includes the adjoint and the trivial representation.

Real and pseudoreal representations

Definition 9.4. A unitary representation V is real if $V \simeq \overline{V}$ by a unitary change of basis.

In this case $\mathcal{T} = \mathcal{T}^*$ is a real anti-symmetric matrix. In this case, we can sensibly restrict $V \simeq \mathbb{R}^n$, i.e. $V \simeq \mathbb{C}^n$, while irreducible as a representation of $\mathfrak{g}_\mathbb{C}$, becomes reducible as a representation of the real $\mathfrak{g}$. If a representation is real then $V \otimes V \supset \mathbf{1}$, i.e. V admits an invariant bilinear form.

This prompts a question: when does an irrep admit an invariant bilinear form? Suppose such

[39] This fails for non-compact LGs and their LAs.

[40] Recall that the study of how these tensor products decompose into irreps goes by the name of "plethysms."

a form exists, i.e.

$$B(v,w) = {}^t v B w \tag{35}$$

for some non-zero matrix B. Invariance then implies

$${}^t\mathcal{T} B + B\mathcal{T} = -\mathcal{T}^* B + B\mathcal{T} = 0 \ , \qquad -\mathcal{T} B^* + B^*\mathcal{T}^* = 0 \ . \tag{36}$$

Exercise 9.5. Use Schur's lemma and (36) to show $\ker B = 0$ and $B^* B = \lambda \mathbb{1}$. Show that any two bilinear invariants B and B' must be related by $B' = \lambda B$.

Without loss of generality we can set $|\lambda| = 1$.

Exercise 9.6. Use (36) to further show that ${}^t B = \pm B$ and $B^* B = \pm \mathbb{1}$ (with the signs correlated, of course).

Thus, we have just two cases to consider: B is either symmetric or anti-symmetric unitary matrix. Since B is invertible, the second case requires $\dim V = 2m$. But now, using the results of Lemmas 1.15 and 1.14 , we see that by a unitary change of basis we have $B = \mathbb{1}$ in the symmetric case and

$$B = J = \begin{pmatrix} 0 & \mathbb{1}_m \\ -\mathbb{1}_m & 0 \end{pmatrix} \ .$$

J is a symplectic structure on the representation. Note that $J^2 = -\mathbb{1}$. So, we have three exclusive possibilities for a non-trivial irrep V.

1. V is *real* if and only if it admits a symmetric invariant bilinear form. In this case $V \simeq \overline{V} \simeq V^\vee$,$\mathcal{T}^* = \mathcal{T}$, and $\wedge^2 V \supset \text{adj}(\mathfrak{g})$, while $\text{Sym}^2 V \supset \mathbf{1}$.

2. V is *pseudoreal* if and only if it admits an anti-symmetric invariant bilinear form. In this case V and $V^\vee \simeq \overline{V}$ are still equivalent by a non-unitary similarity transformation, since $\mathcal{T}^* = J\mathcal{T}J^{-1}$. We have $\wedge^2 V \supset \mathbf{1}$ and $\text{Sym}^2 V \supset \text{adj}(\mathfrak{g})$.

3. V is *complex* if and only if $B = 0$. In this case there is no similarity transformation relating V and $V^\vee$: they are distinct representations, and we just have $V \otimes V^\vee \supset \text{adj}(\mathfrak{g}) \oplus \mathbf{1}$.

Example 9.7. Consider $\mathfrak{g} = \mathfrak{so}(n)$. The fundamental n-dimensional representation V is real. Recall that $\wedge^2 V$ is the adjoint representation of $\mathfrak{so}(n)$. The only simple LA for which $\wedge^2 V = \text{adj}(\mathfrak{g})$ for some representation V is $\mathfrak{so}(n)$.

Next, suppose we take $\mathfrak{g} = \mathfrak{sp}(n)$. The fundamental $2n$-dimensional representation V is pseudo-real. Recall that $\text{Sym}^2 V$ is the adjoint representation of $\mathfrak{sp}(n)$. The only simple LA for which $\text{Sym}^2 V = \text{adj}(\mathfrak{g})$ for some representation V is $\mathfrak{sp}(n)$.

Finally, let $\mathfrak{g} = \mathfrak{su}(n)$. The fundamental n-dimensional representation V is complex. Recall that $V \otimes \overline{V} = \text{adj}(\mathfrak{su}(n)) \oplus \mathbf{1}$. The only simple LA for which $V \otimes \overline{V} = \text{adj}(\mathfrak{g}) \oplus \mathbf{1}$ for some V is $\mathfrak{su}(n)$.

The index of representation

We explored the notion of a bilinear form in full generality above. In fact, we met our first bilinear form long ago: the Killing form is a symmetric bilinear form on the adjoint representation! The condition that may have seemed a little mysterious when we introduced it, namely (9)

$$B([z,x],y) + B(x,[z,y]) = 0 \ ,$$

is nothing but the condition that B is invariant. How many different Killing forms are there? Another application of Schur's lemma yields the answer. Given any other invariant B', $B^{-1}B'$ is an invariant that commutes with all generators, i.e. is a $\mathfrak{g}$-linear map $\mathfrak{g} \to \mathfrak{g}$, and hence is proportional to identity: $B' = kB$ for some constant k.

On the other hand, for any representation V we can define, by analogy with the Killing form

$$B_V(x,y) = \mathrm{Tr}_V\{\rho_V(x)\rho_V(y)\} \ .$$

This is clearly independent of the choice of basis on V, and by the above it satisfies $B_V(x,y) = k_V B(x,y)$ for some constant k_V.

Definition 9.8. The quadratic Casimir of a representation V is defined by $C_V = \frac{\dim \mathfrak{g}}{\dim V} k_V$.

Clearly $C_{\mathrm{adj}} = 1$.

The nicest normalization

So far we have not made any choice of normalization for the Killing form of a simple LA. On the other hand, we just saw that up to normalization this form is unique. The freedom of picking the normalization has an important physical significance: in classical Yang-Mills theory it is the freedom to pick the gauge coupling as we like.

Dynkin described a very elegant and convenient normalization choice for the Killing form of a simple LA $\mathfrak{g}$. Recall that B yields the metric b on the Cartan subalgebra and its dual. Let θ be the longest root of $\mathfrak{g}$. Let B_2 be the Killing form with normalization chosen so that θ has length-squared 2, i.e. $b_2(H_\theta, H_\theta) = 2$. Then for any representation

$$B_V(x,y) = \ell_V B_2(x,y) \ , \tag{37}$$

where $\ell_V \in \mathbb{Z}_{\geq 0}$ is the *(Dynkin) index of representation.*

The index of representation satisfies a number of nice properties. If $V = \Gamma_\lambda$ is an irrep with highest weight λ, then

$$\ell_\lambda = \frac{\dim \Gamma_\lambda}{\dim \mathfrak{g}} b_2(\lambda, \lambda + 2\delta) = \frac{1}{\mathrm{rank}\, \mathfrak{g}} \sum_{\mu \in \mathrm{weights}(\Gamma_\lambda)} b_2(\mu,\mu) \ . \tag{38}$$

Here δ is the Weyl vector defined in the previous lecture (defn. 8.6). The first equality is derived in Cahn following standard quadratic Casimir presentation; the second equality is also non-trivial: it was obtained by Patera et. al. — see Slansky for more information. In particular,

$$\ell(\mathrm{adj}\, \mathfrak{g}) = b_2(\theta, \theta + 2\delta) = 2h^\vee(\mathfrak{g}) \ , \tag{39}$$

where $h^\vee(\mathfrak{g})$ is known as the *dual Coxeter number*. We will not use that term in what follows, but perhaps it's good to know. In the following table we list for every simple LA the indices of representation for the adjoint, as well as the fundamental irrep—the smallest non-trivial representation.[41]

	$\mathfrak{su}(n)$	$\mathfrak{so}(n)$	$\mathfrak{sp}(n)$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{f}_4$	$\mathfrak{g}_2$
$\frac{1}{2}\ell(\text{adj})$	n	$n-2$	$n+1$	12	18	30	9	4
$\ell(\text{fund})$	1	2	1	6	12	60	6	2
$\dim(\text{fund})$	n	n	$2n$	27	56	248	26	7

The index of representation has many uses. Perhaps the simplest is that, in addition to dimension, it is a simple characteristic of a representation, and like the dimension, it has simple behavior under tensor products. It is easy to see from the definition of the trace that

$$\ell(V \oplus W) = \ell(V) + \ell(W) \ , \qquad \ell(V \otimes W) = \ell(V) \dim W + \dim V \ell(W) \ .$$

With a little more work we can also show

$$\ell(\wedge^k V) = \binom{\dim V - 2}{k-1} \ell(V) \qquad \ell(\operatorname{Sym}^k V) = \binom{\dim V + k}{k-1} \ell(V) \ .$$

As such, we can sometimes use it to distinguish representations. For instance, for $\mathfrak{su}(3)$ the irreps $[2,1]$ and $[4,0]$ both have $\dim V = 15$, but $\ell([2,1]) = 20$, while $\ell([4,0]) = 35$.

The index of representation also makes a nice appearance in familiar quantities like the β function of Yang-Mills theories. For instance, for $N = 1$, $d = 4$ SYM with simple gauge algebra $\mathfrak{g}$ and chiral matter $\oplus_i V_i$, the β function for the coupling takes the form

$$\mu \frac{\partial}{\partial \mu} g = -\frac{b}{16\pi^2} g^3 + O(g^5) \ ,$$

where

$$b = \frac{3}{2} \ell(\operatorname{adj} \mathfrak{g}) - \frac{1}{2} \sum_i \ell(V_i) \ . \tag{40}$$

[41]To be precise here, we need to recall the equivalences $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{so}(6) = \mathfrak{su}(4)$ and $\mathfrak{so}(5) = \mathfrak{sp}(2)$. In the table above the $\mathfrak{so}(2k)$ and $\mathfrak{so}(2k-1)$ are to be taken with $k > 3$.

10 Higher order invariants of $\mathfrak{g}$ and anomalies

As we learned repeatedly, the Killing form is an incredibly useful $\mathfrak{g}$-invariant symmetric tensor that we can define for any LA. This tensor is unique up to normalization, or equivalently the choice of representation we use to define the trace. The normalization is determined by a non-negative integer — the index of representation. LAs turn out to have a rich structure of higher rank invariant tensors sometimes known as higher order Casimir operators. It is beyond the scope of this presentation to give a complete overview of these objects, but in this lecture we will try to get a bit of the flavor for them by considering their application to an important problem in high energy physics: the structure of local anomalies in gauge theories.

Anomalies in gauge theory

Consider a theory of Weyl fermions ψ in $\mathbb{R}^{1,2k-1}$ coupled to a background Yang-Mills gauge field A with a classical gauge-invariant action $S_0(\psi, A)$. By integrating out the fermions we obtain an effective action for the background gauge field $S_{\text{eff}}(A)$. One of the most influential mathematical developments in high energy physics in the 60's and 70's was the realization that in general S_{eff} is not gauge-invariant for any choice of local counter-terms. This means that the corresponding gauge theory, where A is dynamical is an inconsistent theory.[42]

The basic structure turns out to be easy to describe. We consider a gauge theory for a compact gauge group G with a simple LA $\mathfrak{g}$ and ψ_L and ψ_R complex Weyl fermions that transform in representations V and W of $\mathfrak{g}$. The gauge theory has a local gauge anomaly if and only if

$$\mathcal{A} = \operatorname{Tr}_V F^{k+1} - \operatorname{Tr}_W F^{k+1} \neq 0$$

for some $F \in \mathfrak{g}$.[43] Note the relative minus between contributions of the left-moving and right-moving fermions.

Consequences of CPT

In a local unitary QFT fermions are space time spinors, and hence their properties depend on the space time dimensionality. We will explore this further in the next few lectures, but for now we have the following distinction.

1. $k = 2m + 1$ implies ψ_L and ψ_R are independent. We say a gauge theory is chiral if $V \neq W$. $\operatorname{Tr}_V F^{2m+2}$ has a definite sign independent of V, and hence any theory with W a trivial representation and V non-trivial is necessarily anomalous. To verify the assertion of the definite sign we recall that F is an anti-Hermitian matrix in every representation, and hence iF has real eigenvalues and can be simultaneously diagonalized in every representation. As a result $\operatorname{sign} \operatorname{Tr}_V F^{2m+2} = (-1)^{m+1}$.

[42]See Weinberg, Vol II, Green, Schwarz, Witten, 13, or Polchinski 12 for some excellent discussions of these topics; the last two references describe the role of anomalies in the construction of consistent string theories.

[43]We will follow the standard abuse of notation here: F is an element of the LA, which we identify with the adjoint representation. Moreover, the representation is assumed to be clear from context. For instance, $\operatorname{Tr}_V F^{k+1}$ should really be written as $\operatorname{Tr} \rho_V(F)^{k+1}$, where $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is the reprsentation.

2. $k = 2m$ implies that ψ_L and ψ_R are CPT conjugate with $W = \overline{V}$, and we say a gauge theory is chiral if $W \neq \overline{V}$. Observe that

$$\mathrm{Tr}_{\overline{V}}\, F^{2m+1} = \mathrm{Tr}_V (F^*)^{2m+1} = \mathrm{Tr}_V (F^\dagger)^{2m+1} = -\,\mathrm{Tr}_V\, F^{2m+1} \ .$$

The first equality follows by the definition of the conjugate representation; the second by $\mathrm{Tr}\, X = \mathrm{Tr}^t\, X$, and the last by anti-Hermiticity of the generators. This means we can simplify $\mathcal{A} = 2\,\mathrm{Tr}_V\, F^{2m+1}$.

Lie algebra structure of $\mathrm{Tr}_V\, F^n$

So, to check if our gauge theory is anomalous, we should study these traces. A fundamental property of these objects is that they are $\mathfrak{g}$-invariant. That is, under $\delta_\epsilon F = [F, \epsilon]$, we have

$$\delta_\epsilon\, \mathrm{Tr}_V\, F^n = n\, \mathrm{Tr}_V\, [F,\epsilon] F^{n-1} = n\, \mathrm{Tr}_V (F\epsilon - \epsilon F) F^{n-1} = 0 \ .$$

By expanding in generators we obtain

$$\mathrm{Tr}_V\, F^n = F_{A_1} F_{A_2} \cdots F_{A_n} \underbrace{\mathrm{Tr}_V\, \mathcal{T}^{(A_1} \mathcal{T}^{A_2} \cdots \mathcal{T}^{A_n)}}_{\equiv (\mathcal{D}_V^n)^{A_1 \cdots A_n}} \ .$$

We see that $\mathcal{D}_V^n$ is a totally symmetric rank n tensor on $\mathfrak{g}$, i.e. $\mathcal{D}_V^n \in \mathrm{Sym}^n(\mathrm{adj}\, \mathfrak{g})$, and invariance implies it is a singlet in $\mathrm{Sym}^n(\mathrm{adj}\, \mathfrak{g})$ Of course the Killing form is a simple example of such an object. In that case we saw that the dependence on V is simply through a normalization — the index of the representation. For $n > 2$ $\mathcal{D}_V^n$ have a more elaborate dependence on both $\mathfrak{g}$ and V. So, to have an anomaly for a gauge theory based on $\mathfrak{g}$ we have two necessary conditions:

1. $\mathrm{Sym}^n(\mathrm{adj}\, \mathfrak{g}) \supset \mathbf{1}$ for the appropriate n;
2. $\mathcal{D}_V^n \neq 0$.

We also have the useful relation

$$\mathrm{Tr}_{V_1 \oplus V_2}\, F^n = \mathrm{Tr}_{V_1}\, F^n + \mathrm{Tr}_{V_2}\, F^n.$$

This means that we can consider anomaly contributions from each irrep separately.

A simple consequence of this is that in $d = 4m$ dimensions only fermions in a complex representation make a contribution to $\mathcal{A}$. To see this, consider the fermions belonging to a real or pseudoreal irrep V. Their contribution to the anomaly is

$$\mathcal{A} = 2\,\mathrm{Tr}_V\, F^{2m+1} = 2\,\mathrm{Tr}_V (S^{-1} F^* S)^{2m+1} = -\,\mathrm{Tr}_V\, F^{2m+1} = 0.$$

In the first equality we used the result that F^* is similar to F when V is real or pseudoreal. This should be contrasted with $d = 4m+2$ dimensions, where real and pseudoreal representations can also make interesting contributions to $\mathcal{A}$.

Anomalies in even dimensions

Let's now see how the story works out in some familiar dimensions. We start with $d = 2$, where

$$\mathcal{A} = \mathrm{Tr}_V F^2 - \mathrm{Tr}_W F^2 = [\ell(V) - \ell(W)]\, B_2(F, F) \ .$$

In this case the index of representation determines the anomaly structure.

Moving on to $d = 4$ dimensions, the only $\mathfrak{g}$ that can have anomalies are $\mathfrak{g} = \mathfrak{su}(n + 1)$ with $n > 1$. This is based on two facts:

1. the only $\mathfrak{g}$ with complex representations are $\mathfrak{so}(4n + 2)$, $\mathfrak{su}(n + 1)$ for $n > 1$ and $\mathfrak{e}_6$.
2. $\mathcal{D}^3_V = 0$ for $\mathfrak{so}(4n + 2)$ and $\mathfrak{e}_6$.

Exercise 10.1. Use your favorite LA computer program to check that $\mathcal{D}^3_V = 0$ for $\mathfrak{so}(10)$ and $\mathfrak{e}_6$. Hint: study the decomposition of $\mathrm{Sym}^3(\mathrm{adj}(\mathfrak{g}))$ for these LAs.

Of course, there is a lot more to say about anomalies. For instance, there can be mixed gauge anomalies when the gauge group is $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and there are also gravitational anomalies. For all of these the symmetric invariant tensors play a key role.

Exercise 10.2. If you have never gone through the exercise, it is highly recommended that you work through anomaly cancellation for the Standard Model.

Invariant polynomials

Fix a simple compact LA $\mathfrak{g}$ and a representation V with generators $\mathcal{T}^A$, $A = 1, \ldots, \dim \mathfrak{g}$.[44] We showed that we can normalize the generators so that $\mathrm{Tr}_V \mathcal{T}^A \mathcal{T}^B = -\delta^{AB}$. We just discussed a large class of invariant symmetric tensors $\mathcal{D}^n_V$ constructed from $\mathrm{Tr}_V F^n$. A seeming generalization of the construction is to take $\mathrm{Tr}_V (F_1 F_2 \cdots F_n)$ for n independent elements $F_i \in \mathfrak{g}$. This is still invariant because we can think of the adjoint action $\delta F_i = [M, F_i]$ as an infinitesimal version of a similarity transformation $F_i \to S F_i S^{-1}$, where $S = \mathbb{1} + M$, and the trace is clearly invariant under this. On the other hand, expanding this in a basis $F_i = \sum_A F_{iA} \mathcal{T}^A$ leads to a more general class of invariant tensors

$$\mathrm{Tr}_V \mathcal{T}^{A_1} \cdots \mathcal{T}^{A_n} : \mathfrak{g}^{\otimes n} \to \mathbb{C} \ .$$

Unlike the $\mathcal{D}^n_V$, these have no symmetry properties. In fact, what seems like a great generalization is not so interesting. The reason for this is that every $\mathfrak{g}$ has, in addition to the Killing form, another basic invariant determined by the Lie bracket. In terms of the generators we have

$$[\mathcal{T}^A, \mathcal{T}^B] = \sum_D C^{AB}{}_C \mathcal{T}^C \ ,$$

where $C^{AB}{}_C = -C^{BA}{}_C$ are the structure constants of $\mathfrak{g}$. The Jacobi identity is equivalent to C being an invariant tensor. This means that we can reduce any invariant tensor to a combination of structure constants and invariant symmetric tensors. So, to describe all invariant tensors it is

[44] The discussion here largely follows [5].

sufficient to describe the symmetric tensors, and as we saw, we can obtain these as $\mathrm{Tr}_V F^n$ for various V and various n. In fact, all symmetric tensors can be obtained by taking a sum over these traces for various V and n — the result is a large polynomial algebra $P_{\mathfrak{g}}$.

It turns out that this presentation is grossly redundant: a theorem of Chevalley shows that $P_{\mathfrak{g}}$ is freely generated by rank $\mathfrak{g}$ elements known as the *primitive invariants* of $\mathfrak{g}$. The ranks of these primitive invariants depend on the algebra:

Lie algebra	ranks of primitive invariants
$A_k\ ,k \geq 1$	$2, 3, \ldots, k+1$
$B_k\ ,k \geq 2$	$2, 4, \ldots, 2k$
$C_k\ ,k \geq 3$	$2, 4, \ldots, 2k$
$D_k\ ,k \geq 4$	$2, 4, \ldots, 2k-2, k$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
G_2	$2, 6$

The ranks of the primitive invariants are sometimes called the exponents of $\mathfrak{g}$.

Relations between the traces

To make this all a bit clearer, we consider a classic example that turns out to play an important role in string theory. We consider some invariant polynomials of $\mathfrak{so}(n)$. Let tr denote the trace in the fundamental n-dimensional representation and Tr the trace in the adjoint.

Exercise 10.3. If F is an $n \times n$ anti-symmetric matrix F_{ac}, i.e. a generator of $\mathfrak{so}(n)$ in the fundamental representation, show that the corresponding element in the adjoint representation is given by

$$F_{ab,cd} = \tfrac{1}{2}(F_{ac}\delta_{bd} - F_{bc}\delta_{ad} - F_{ad}\delta_{bc} + F_{bd}\delta_{ac}) \ .$$

We discover the following identities between various tensors (you should verify these):

$$\begin{aligned} \mathrm{Tr}\, F^2 &= (n-2)\,\mathrm{tr}\, F^2 \ , \\ \mathrm{Tr}\, F^4 &= (n-8)\,\mathrm{tr}\, F^4 + 3(\mathrm{tr}\, F^2)^2 \\ \mathrm{Tr}\, F^6 &= (n-32)\,\mathrm{tr}\, F^6 + 15\,\mathrm{tr}\, F^2\,\mathrm{tr}\, F^4 \ . \end{aligned}$$

So, for instance, for $n = 32$ $\mathrm{Tr}\, F^6$ factorizes into a product of lower rank invariants, even though there is an independent rank 6 invariant given by $\mathrm{tr}\, F^6$.

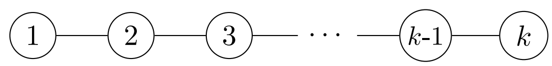
11 Properties of simple LAs

Today we start looking at some details of representations of LAs in the Cartan catalog. There are many questions we might want to know. An incomplete but useful list, which we will try to present for the simple LAs, is the following.

1. It's good to know the rank, dimension, highest weight of the adjoint, and Λ_W/Λ_R, as well as the corresponding simply connected compact Lie group.
2. The basic irreps, with Dynkin labels $[0,\dots,1,\dots,0]$ are the simplest representations and can be used to generate all others via tensor products. It's good to know something about their structure.
3. (Pseudo)reality properties of irreps are very useful.
4. Various sub-algebras and the decomposition of representations with respect to these is a big and important topic. We will say what time allows about it.

The A_k series

These are surely the most familiar simple LAs: $A_k = \mathfrak{sl}_{k+1}\mathbb{C}$. They have the Dynkin diagram



The dimension is of course $\dim = (k+1)^2 - 1$. With this labeling, we define the fundamental representation to be

$$V = \Gamma_{[1,0,\dots,0]} \ .$$

This has $\dim V = k+1$ and $\ell(V) = 1$. The remaining basic irreps are obtained by taking $\wedge$ products of the fundamental, e.g.

$$\wedge^2 V = \Gamma_{[0,1,0,\dots,0]} \ , \qquad \wedge^s V = \Gamma_{[0,\dots,\underbrace{1}_{\text{s spot}},0,\dots,0]} \ .$$

In particular, the anti-fundamental representation is $\overline{V} = [0,\dots,0,1] = \wedge^k V$, and more generally, $\wedge^s V = \wedge^{k+1-s}\overline{V}$.

Once we have the fundamental representation, it is easy to obtain the adjoint. We know on general grounds that $V \otimes \overline{V} \supseteq \text{adj} \oplus \mathbf{1}$, and in particular for A_k $V \otimes \overline{V} = \text{adj} \oplus \mathbf{1}$, while just by adding the highest weights of V and $\overline{V}$ we obtain $V \otimes \overline{V} \supseteq \Gamma_{[1,0,\dots,0,1]}$. Hence,

$$\text{adj} = \Gamma_{[1,0,\dots,0,1]} \ .$$

Generalizing our study of $\mathfrak{sl}_3\mathbb{C}$, it is not too hard to show that $\Lambda_W/\Lambda_R = \mathbb{Z}_{k+1}$. Upon exponentiation this becomes the center of the simply connected compact LG $\mathrm{SU}(k+1)$.

Reality properites of irreps

The fundamental representation is complex for $k > 1$.[45] More generally, a representation $\Gamma_{[a_1,\ldots,a_k]}$ is complex unless

$$[a_1, a_2, \ldots, a_k] = [a_k, a_{k-1}, \ldots, a_1] \ .$$

Let $h = \sum_i a_i$. A representation $\Gamma_{[a_1,a_2,\ldots,a_2,a_1]}$ is real(pseudoreal) if h is even (odd).

A little branching for A_k

In this section we start introducing some concepts in the general study of sub-algebras and decomposition of representations. Let's begin with a familiar example. Everyone knows that $\mathfrak{su}(k+1) \supset \mathfrak{su}(k) \oplus \mathfrak{u}(1)$. The reason this is so "obvious" is because of the simplicity of the fundamental representation. The fundamental of $\mathfrak{su}(k+1)$, which we will denote V^{k+1} consists of traceless anti-Hermitian matrices, which we can decompose as

$$M_{k+1} = \begin{pmatrix} M_k + ia\mathbb{1}_{k\times k} & N \\ -N^\dagger & -kia \end{pmatrix} \ ,$$

where N is a $k \times 1$ matrix, a is a real constant, and M_k is a $(k+1)^2$ anti-Hermitian matrix. Clearly by setting $N = 0$ we obtain the generators of $\mathfrak{su}(k)$—the M_k, and the generator of $\mathfrak{u}(1)$—the matrix proportional to a. Evidently, we have the following branching:

$$\begin{aligned} \mathfrak{su}(k+1) &\supset \mathfrak{su}(k) \oplus \mathfrak{u}(1) \ , \\ V^{k+1} &= (V^k)_{+1} \oplus (\mathbf{1})_{-k} \ . \end{aligned}$$

Note we are following the standard practice of specifying the $\mathfrak{u}(1)$ representation, aka the charge, by a subscript on the representations of the semi-simple factor—in this case $\mathfrak{su}(k)$. Since we saw above that all the basic representations are determined in terms of tensor products of the fundamental, the branching of V^{k+1} can be used to determine the branching of every other irrep of $\mathfrak{su}(k+1)$.[46] For example, we can decompose the adjoint as follows.

$$\begin{aligned} V^{k+1} \otimes \overline{V}^{k+1} &= \mathrm{adj}\, \mathfrak{su}(k+1) \oplus \mathbf{1} \\ &= (V^k_{+1} \oplus \mathbf{1}_{-k}) \otimes (\overline{V}^k_{-1} \oplus \mathbf{1}_{+k}) \\ &= \mathrm{adj}(\mathfrak{su}(k))_0 \oplus \mathbf{1}_0 \oplus V^k_{k+1} \oplus \overline{V}^k_{-k-1} \oplus \mathbf{1}_0 \ , \\ \Longrightarrow \mathrm{adj}(\mathfrak{su}(k+1)) &= \mathrm{adj}(\mathfrak{su}(k))_0 \oplus V^k_{k+1} \oplus \overline{V}^k_{-k-1} \oplus \mathbf{1}_0 \ . \end{aligned}$$

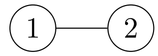
Of course we see this structure explicitly in the decomposition of M_{k+1}.

Let's see how we might think of this decomposition more abstractly in terms of roots and

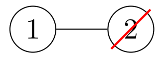
[45] The fundamental of $\mathfrak{su}(2)$ is pseudoreal. This is a basic reason for why A_1 is a rather special member of the A_k series.

[46] This is handy when one needs to decompose irreps by hand; of course computer packages do not need to rely on such tricks: they can follow a more general procedure of decomposing the weights into irreps of the sub-algebra.

weights. We start with the simplest example of $\mathfrak{su}(3) \supset \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Remember that the Dynkin diagram for $\mathfrak{su}(3)$ is



It represents the simple roots of $\mathfrak{su}(3)$, and each simple root generates an $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ sub-algebra. For instance, we might want to consider the $\mathfrak{su}(2)$ generated by the first simple root. So, all we need to do is "forget" about the second simple root. We do this by crossing out the corresponding node:



Once we drop the second simple root, we are left with the $\mathfrak{su}(2)$ corresponding to the first root, but we still have the full two-dimensional Cartan subalgebra of $\mathfrak{su}(3)$ with basis H_{α_1} and H_{α_2} — that is of course the Cartan of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. We can decompose the weights of any irrep of $\mathfrak{su}(3)$ according to these, but there is one complication: we want to pick a basis so that all weights in a given $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ representation have the same $\mathfrak{u}(1)$ charge. In this case we pick $H = H_{\alpha_1} + 2H_{\alpha_2}$ as the generator of $\mathfrak{u}(1)$. Before we explain the choice, let's see that it achieves the expected result. Recall that for a weight $\lambda = [a_1, \ldots, a_n]$ we have $\lambda(H_{\alpha_i}) = a_i$. Hence, when we decompose the fundamental of $\mathfrak{su}(3)$, we have

$\Gamma_{[10]}$	$\mathfrak{su}(3)$	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$
	$[1\ \ 0]$	$[1]_{+1}$
	$[-1\ 1]$	$[-1]_{+1}$
	$[0\ -1]$	$[0]_{-2}$

We see that taking $H = H_{\alpha_1} + 2H_{\alpha_2}$ does the trick: the weights of both states in the fundamental of $\mathfrak{su}(2)$ have the same $\mathfrak{u}(1)$ charge, and we obtain the expected $\mathbf{3} = \mathbf{2}_1 \oplus \mathbf{1}_{-2}$ decomposition.

Now let us explain the trick in more generality. Whenever we take a LA of $\mathfrak{g}$ and simply drop a simple root α_k from the Dynkin diagram, we find a subalgebra $\mathfrak{h} \oplus \mathfrak{u}(1) \subset \mathfrak{g}$ consisting of the LA generated by the remaining simple roots and the Cartan generator associated to the dropped root. We wish to pick the $\mathfrak{u}(1)$ generator in such a way that it commutes with the $\mathfrak{h}$ generators. Since the $\mathfrak{h}$ action on a representation is generated by the remaining roots α_i, $i \neq k$, it is sufficient to make sure that the $\mathfrak{u}(1)$ generator H assigns charge 0 to α_i for $i \neq k$.[47]

That is just what $H = H_{\alpha_1} + 2H_{\alpha_2}$ achieves in our example. From the Cartan matrix we have

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} , \tag{41}$$

where the w_i are the fundamental weights satisfying $w_i(H_{\alpha_j}) = \delta^i_j$. Hence $\alpha_1(H_{\alpha_1} + 2H_{\alpha_2}) = 0$ as desired.

[47] A nice way to check the $\mathfrak{u}(1)$ charges is to remember that all original generators were traceless; hence, for any decomposition of a representation the sum over all states weighted by the $\mathfrak{u}(1)$ charge must be 0.

As another example, let's consider $\mathfrak{su}(4) \supset \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ obtained from

Exercise 11.1. To check that you remember how to unravel the simple roots/fundamental weights definitions, show that

$$\alpha_i(H_{\alpha_k}) = C_{ik} \ ,$$

where C is the Cartan matrix.

Using the exercise and the Cartan matrix of $\mathfrak{su}(4)$

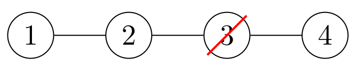
$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \ ,$$

we see that $H = H_{\alpha_1} + 2H_{\alpha_2} + H_{\alpha_3}$ satisfies $\alpha_i(H) = 0$ for $i = 1, 3$. We now apply this to the fundamental:

$\Gamma_{[100]}$	$\mathfrak{su}(3)$	$\mathfrak{su}(2)^{\oplus 2} \oplus \mathfrak{u}(1)$
	$[1, 0, 0]$	$[1; 0]_{+1}$
	$[-1, 1, 0]$	$[-1; 0]_{+1}$
	$[0, -1, 1]$	$[0; 1]_{-1}$
	$[0, 0, -1]$	$[0; -1]_{-1}$

We used the semi-colon to distinguish the fundamental weights of the two $\mathfrak{su}(2)$s.

Exercise 11.2. Use the same technology to work out the $\mathfrak{su}(5) \supset \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ decomposition according to



Obtain the decompositions

$$\mathbf{5} = (\mathbf{3}, \mathbf{1})_{-2} \oplus (\mathbf{1}, \mathbf{2})_{+3} \ ,$$
$$\mathbf{10} = (\mathbf{3}, \mathbf{2})_{+1} \oplus (\overline{\mathbf{3}}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{1})_{6} \ .$$

Compare $\overline{\mathbf{5}} \oplus \mathbf{10}$ with the Standard Model spectrum of massless fermions (without massive neutrinos!). Read one of the greatest physics papers ever [6] .

Beginnings of D_k

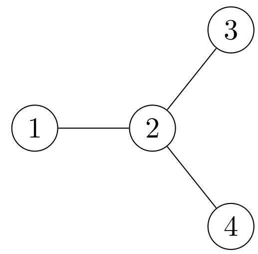
Having said quite a bit about the A_k series, we are now ready to move on to $D_k = \mathfrak{so}(2k)$. The first two entries, $D_1 = \mathfrak{so}(2)$, $D_2 = \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, are a bit boring: they are not simple in the technical sense and very simple in the colloquial sense. So, we move on to $D_3 = \mathfrak{so}(6) = \mathfrak{su}(4)$.

The last equality means that we already know everything about its representations as well, but it's good to think about the basic representations from the $\mathfrak{so}(6)$ perspective. We have:

$$\Gamma_{[100]} = \mathbf{4}\ ,\quad \ell(\mathbf{4}) = 1\ ; \qquad \Gamma_{[001]} = \overline{\mathbf{4}}\ ,\quad \ell(\overline{\mathbf{4}}) = 1\ ; \qquad \Gamma_{[010]} = \mathbf{6}\ ,\quad \ell(\overline{\mathbf{6}}) = 2\ .$$

The first two, from the point of view of $\mathfrak{so}(6)$ correspond to the two Weyl spinor representations, while the last one is the "fundamental" of $\mathfrak{so}(6)$ — a good check of sensibility is that it is indeed a real representation (see above!). Clearly no tensor product of $\mathbf{6}$ can lead to a $\mathbf{4}$ or $\overline{\mathbf{4}}$, since every entry in the decomposition will be manifestly real.

Next, we move on to the very interesting case of $D_4 = \mathfrak{so}(8)$. This has the Dynkin diagram



The diagram has an obvious S_3 permutation symmetry of the nodes $1, 3, 4$. In fact, it is the only diagram that has a $\mathbb{Z}_3$ factor in its automorphism group. The $\mathbb{Z}_3$ action is known as "triality," and it has some remarkable consequences. First, let's describe the basic representations:

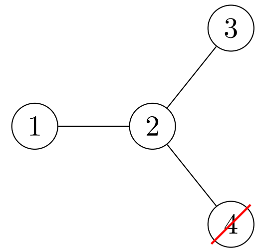
$$\begin{aligned} V = \Gamma_{[1000]} &= \mathbf{8}^{\boldsymbol{v}}\ , && \text{is the fundamental} \\ \wedge^2 V = \Gamma_{[0100]} &= \mathbf{28}\ , && \text{is the adjoint} \\ \Gamma_{[0001]} &= \mathbf{8}^{\boldsymbol{s}}\ ,\quad \Gamma_{[0010]} = \mathbf{8}^{\boldsymbol{c}}\ , && \text{are the Weyl spinors}\ . \end{aligned}$$

As for $\mathfrak{so}(6)$, the spinor irreps cannot be obtained from the fundamental. The dimensions of the spinor irreps are in this case fixed by triality: they must be the same as the dimension of the fundamental; similarly, they are both real irreps, and $\ell(\mathbf{8}^{\boldsymbol{v},\boldsymbol{c},\boldsymbol{s}}) = 2$. Some additional properties are

$$\begin{aligned} \mathbf{8}^{\boldsymbol{i}} \otimes \mathbf{8}^{\boldsymbol{i}} &= \mathbf{1} \oplus \mathrm{Sym}^2\, \mathbf{8}^{\boldsymbol{i}} \oplus \mathrm{adj}\ , \\ \mathbf{8}^{\boldsymbol{i}} \otimes \mathbf{8}^{\boldsymbol{j}} &= \mathbf{8}^{\boldsymbol{k}} \oplus \mathbf{56}^{\boldsymbol{v}}\ , \end{aligned}$$

where $i, j, k = v, s, c$, the i, j, k in the second decomposition are cyclic, and $\mathrm{Sym}^2\, \mathbf{8}^{\boldsymbol{v}} = \Gamma_{[2000]}$, $\mathbf{56}^{\boldsymbol{v}} = \Gamma_{[0011]}$. We can apply our previous technology again to study some simple branching.

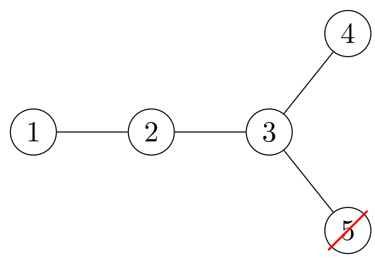
Exercise 11.3. Consider $\mathfrak{so}(8) \supset \mathfrak{su}(4) \oplus \mathfrak{u}(1)$ that is constructed by striking the last node:



Show that $H = H_{\alpha_1} + 2H_{\alpha_2} + H_{\alpha_3} + 2H_{\alpha_4}$ is a good generator for $\mathfrak{u}(1)$ and determine the decomposition

$$\mathbf{8}^{\boldsymbol{v}} = \mathbf{4}_{+1} \oplus \overline{\mathbf{4}}_{-1} \ .$$

Exercise 11.4. Using the same tricks, consider the decomposition $\mathfrak{so}(10) \supset \mathfrak{su}(5) \oplus \mathfrak{u}(1)$



Show that $\mathbf{16} = \Gamma_{[00010]}$ has the decomposition

$$\mathbf{16} = \overline{\mathbf{5}}_{-3} \oplus \mathbf{10}_{-1} \oplus \mathbf{1}_{-5} \ .$$

Using exercise 11.2 and the results of the previous lecture, conclude that the Standard Model spectrum is anomaly-free.[48]

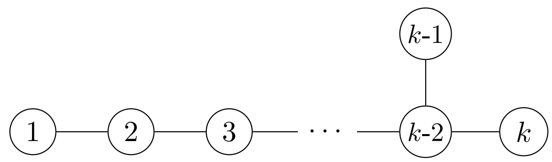
[48]There is one little loophole here: in our discussion of anomalies we did not explicitly talk about mixed gravitational-gauge anomalies. Fortunately, in four dimensions these are very simple: they vanish if and only if the sum of U(1) charges of left-moving fermions vanishes for every U(1) factor. Once the SM fermion content is embedded in SO(10) as suggested here, we don't need to worry about this mixed anomaly.

12 Properties of simple LAs, II

We continue the discussion of the Cartan catalog begun in the last lecture. Our first job is to complete the tale of the D_n series

Finishing up the D_k series

Last time we discussed the properties of D_4, which is distinguished from all other simple LAs by the remarkable property of triality. More generally we have the D_k with Dynkin diagram



(42)

The diagram has a $\mathbb{Z}_2$ symmetry exchanging the last two nodes, leading to an outer automorphism of the LA. The basic representations are

$$V = \Gamma_{[1,0,0,\ldots,0]} \ , \qquad\qquad \wedge^s V = \Gamma_{[0,\ldots,\underbrace{1}_{\text{s spot}},0,\ldots,0]} \ , s \le k-2 \ .$$

The fundamental V is real, has dim $= 2k$ and $\ell(V) = 2$. The adjoint is of course adj $= \wedge^2 V = [0,1,0,\ldots,0]$. It is interesting to consider what happens to higher wedge powers. We already discussed these issues some time ago — see below Exercise 3.14 , but let's summarize the details: the fundamental representation has two invariant tensors: the usual Kronecker δ (aka the invariant symmetric bilinear) and the totally anti-symmetric rank $2k$ tensor ϵ (aka the top power $\wedge^{2k}V$). The former is responsible for V being real, while the latter yields an isomorphism $\wedge^s V = \wedge^{2k-s}V$. So, to complete the $\wedge^s V$ catalog it is sufficient to list the two remaining values: $s = k-1$ and $s = k$. We have

$$\wedge^{k-1}V = \Gamma_{[0,\ldots,0,1,1]} \ , \qquad\qquad \wedge^k V = \underbrace{\Gamma_{[0,\ldots,0,2,0]}}_{(\wedge^k V)_+} \oplus \underbrace{\Gamma_{[0,\ldots,0,2]}}_{(\wedge^k V)_-} \ . \tag{43}$$

Finally, the list of basic representations is completed by the Weyl spinor representations:

$$W = [0,\ldots,0,1,0] \ , W' \qquad\qquad = [0,\ldots,0,0,1] \ . \tag{44}$$

These both have dim $= 2^{k-1}$ and $\ell(W) = \ell(W') = 2^{k-3}$.

Reality

The fundamental is always real, as are all of its tensor products. The reality properties of the other irreps depend on k. We have the following break-down.

1. D_{4k} has only real irreps.

2. D_{2k+1} can have complex irreps:

$$[a_1, \ldots, a_{2k-1}, a_{2k}, a_{2k+1}] \begin{cases} \text{is real} & \text{if } \; a_{2k} = a_{2k+1} \\ \text{complex} & \text{otherwise} \end{cases} .$$

   In this case the spinor irreps are complex conjugates of each other: $W' = \overline{W}$.

3. D_{4k+2} irreps are self-conjugate:

$$[a_1, \ldots, a_{4k}, a_{4k+1}, a_{4k+2}] \begin{cases} \text{is real} & \text{if } \; a_{4k+1} = a_{4k+2} \\ \text{pseudoreal} & \text{otherwise} \end{cases} .$$

Associated compact LGs

The D_k series gives us a great opportunity to discuss the relation between LAs and the associated LGs. We keep the discussion informal, as we do not have time to go in great detail, but we hope to provide some statements useful for a first pass. Much more information can be found in Fulton& Harris.

Let $\mathfrak{g}$ be a simple LA and $\mathfrak{g}_{\mathbb{R}}$ be an associated compact form whose exponentiation yields a compact Lie group G.[49] The exponential map exp; $\mathfrak{g}_{\mathbb{R}} \to G$, when restricted to the Cartan subalgebra $\mathfrak{g}_{\mathbb{R}}$ yields a map

$$\exp(2\pi i \, \mathfrak{h}_{\mathbb{R}}) \simeq T \subset \mathfrak{g} \;\; , \tag{45}$$

where $T \simeq (S^1)^{\operatorname{rank} \mathfrak{g}}$ is the Cartan torus.[50]

Compactness requires that the Cartan torus is really that — there are identifications on the elements of $\mathfrak{h}_{\mathbb{R}}$ Suppose we have some non-zero $H \in \mathfrak{h}$ satisfying $e^{2\pi i H} = 1$. Now consider some representation V with a weight decomposition $V = \bigoplus_\beta V_\beta$. The identification is compatible with the weights of V if and only if

$$V_\beta = e^{2\pi i H} V_\beta = e^{2\pi i \beta(H)} V_\beta \implies \beta(H) \in \mathbb{Z} \; .$$

If this does not hold, then V, while being a representation of $\mathfrak{g}$ is not a representation of G. To keep all irreps of $\mathfrak{g}$ we demand that $\beta(H) \in \mathbb{Z}$ for all weights $\beta \in \Lambda_W$ and all H such that $e^{2\pi i H} = 1$. This means

$$H \in \Gamma_R = \left\{ \sum_i m_i H_{\alpha_i} \;\; | \;\; m_i \in \mathbb{Z} \right\} \; . \tag{46}$$

Γ_R is known as the co-root lattice. This leads to the Cartan torus $T = \mathfrak{h}_{\mathbb{R}} / \Gamma_R$. As the following theorem shows, this is the "biggest" torus we can get.

Theorem 12.1. *The Lie group with $T = \mathfrak{h}_{\mathbb{R}} / \Gamma_R$, which we will call $\widetilde{G}$, is simply connected and has center $Z(\widetilde{G}) = \Lambda_W / \Lambda_R$. The center elements act by roots of unity on the irreps. All other*

[49] That is, the complexification of $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{g}$, and exponentiation of $\mathfrak{g}_{\mathbb{R}}$ produces a compact Lie Group.
[50] There is an obvious associated algebraic object: $\exp(2\pi i \, \mathfrak{h}) = (\mathbb{C}^*)^{\operatorname{rank} \mathfrak{g}}$.

compact LGs with Lie Algebra $\mathfrak{g}$ are obtained by quotients of $\widetilde{G}$ by a subgroup of the center $Z' \subset Z$; this keeps the irreps invariant under Z'.

Proof. The proofs of these statements are given in Fulton&Harris. □

So, there is a compact LG with representations in 1:1 correspondence with representations of $\mathfrak{g}$ for every simple LA. There are other compact LGs with same LA that have a smaller Cartan torus and miss some of the irreps of $\mathfrak{g}$. A lot more can be said here, and to complete the story we could also discuss various non-compact LGs associated to the simple LAs, as well as their irreps. One cannot do everything, so we must leave the reader to pursue this topic further in the references. Fulton&Harris is a good place to start.

Let us apply this wisdom to the D_k. For these we have

$$\Lambda_W/\Lambda_R = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k \text{ even} \\ \mathbb{Z}_4 & k \text{ odd} \end{cases}$$

The simply connected compact LGs associated to the D_k are known as $\mathrm{Spin}(2k)$. We already met some of them: $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and $\mathrm{Spin}(6) = \mathrm{SU}(4)$. More generally the spin groups are more interesting and admit the following quotients:

$$\begin{array}{ccccc} & & \mathrm{Spin}(4k) & & Z = \mathbb{Z}_2^1 \times \mathbb{Z}_2^2 \\ & {}^{/Z_2^1}\swarrow & & \searrow^{/\mathbb{Z}_2^2} & \\ \mathrm{SO}(4k) & & & & \mathrm{Spin}(4k)/\mathbb{Z}_2 \end{array}$$

The action of the center of $\mathrm{Spin}(4k)$ on the representations is as follows. The non-trivial elements act as

$$\begin{array}{cccc} \mathbb{Z}_2^1 & V \to V & W \to -W & W' \to -W' \\ \mathbb{Z}_2^2 & V \to -V & W \to -W & W' \to W' \end{array}$$

So, we see that $SO(4k)$ and $\mathrm{Spin}(4k)/\mathbb{Z}_2$ both have $\pi_1 = \mathbb{Z}_2$ and $Z = \mathbb{Z}_2$, but while the former has no spinor representations, the latter has no "vector" (aka fundamental) representations, nor any W irreps.[51] The reader may have encountered the second of the groups in the study of the heterotic string, where both $\mathrm{Spin}(16)/\mathbb{Z}_2$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ play a prominent role.

For D_{2k+1} the situation is a little more "boring". We have $Z = \mathbb{Z}_4$ generated by $\zeta = i$, with the action

$$\begin{array}{cccc} \mathbb{Z}_4 & V \to -V & W \to iW & W' \to -iW' \end{array}$$

[51] It should be clear that the $\mathbb{Z}_2$ action also projects out other irreps. For instance, $V \otimes V \otimes V$ is projected out from $\mathrm{Spin}(4k)/\mathbb{Z}_2$, while $V \otimes V$ is kept. The latter had better be true, since it contains the adjoint representation!

To obtain the familiar $\mathrm{SO}(4k+2)$ we quotient by $\mathbb{Z}_2 \subset Z$ generated by ζ^2, which projects out the spinor representations.

An important caveat to this story is that $g_1 \subset g_2 \;\not\Longrightarrow \widetilde{G}_1 \subset \widetilde{G}_2$. Here are two misquotations commonly found in string literature:

$$\mathfrak{so}(16) \subset \mathfrak{e}_8 \qquad \mathrm{Spin}(16) \not\subset \mathrm{E}_8 \qquad \mathrm{Spin}(16)/\mathbb{Z}_2 \subset \mathrm{E}_8 \tag{47}$$

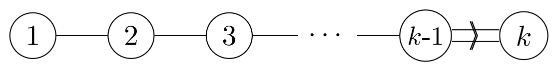
$$\mathfrak{su}(3) \oplus \mathfrak{e}_6 \subset \mathfrak{e}_8 \qquad \mathrm{SU}(3) \times \mathrm{E}_6 \not\subset E_8 \qquad (\mathrm{SU}(3) \times \mathrm{E}_6)/\mathbb{Z}_3 \subset \mathrm{E}_8 \tag{48}$$

The left and right column are true statements; the middle is a common misquotation. Of course it does not matter if one is just talking about Lie algebras, but one should keep the correct group statement in mind.

Remaining classical LAs

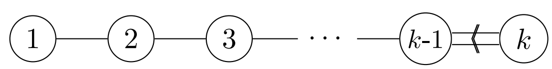
We have two more infinite series of LAs to consider: the B_k and the C_k. Unlike the A and D series, these have roots of different lengths.

The $B_k = \mathfrak{so}(2k+1)$ have Dynkin diagram



The basic representations consist of the fundamental $V = \Gamma_{[1,0,\ldots,0]}$ of $\dim = 2k+1$ and $\ell(V) = 2$, and the $\wedge^s V$ for $s = 2, \ldots, k-1$, with $\wedge^2 V = \Gamma_{[0,1,0,\ldots,0]}$ being the adjoint, and, more generally, $\wedge^s V$ having the 1 in the s-th spot. We also have $\wedge^k V = [0, \ldots, 0, 2]$ and $\wedge^s V = \wedge^{2k+1-s} V$. Finally there is the spinor irrep $W = \Gamma_{[0,\ldots,0,1]}$ with $\dim = 2^k$ and $\ell(W) = 2^{k-2}$. All irreps of B_k are real and $\Lambda_W/\Lambda_R = \mathbb{Z}_2$. This gives us the covering of LGs $\mathrm{Spin}(2k+1) \to \mathrm{SO}(2k+1)$, where the latter has no spinor irrep. There is a LG coincidence that $\mathrm{Spin}(5) = \mathrm{Sp}(2)$.

The $C_k = \mathfrak{sp}(k)$ series have Dynkin diagram



The fundamental representation $V = \Gamma_{[1,0,\ldots,0]}$ is pseudoreal, has $\dim = 2k$ and $\ell(V) = 1$. Tensors of V generate the remaining basic irreps; in particular $\mathrm{adj} C_k = \mathrm{Sym}^2 V = \Gamma_{[2,0,\ldots,0]}$.

Exercise 12.2. This is another exercise for you to get comfortable with your favorite LA software. Verify that all basic irreps of C_6 can be obtained by taking various tensors of V.

We have $\Lambda_W/\Lambda_R = \mathbb{Z}_2$, and the irrep reality properties are determined by

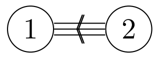
$$[a_1, a_2, \ldots, a_k] \begin{cases} \text{is real} & \text{if } a_1 + a_3 + a_5 + \ldots \text{ is even} \\ \text{is pseudoreal} & \text{otherwise} \end{cases}$$

The exceptional LAs

We now come to the exceptional LAs. The first question we must address is whether these really are LAs. After all, we only built the root systems, and one should ask whether there are generators that realize such an adjoint representation and satisfy the Jacobi identity. The short answer is yes, there are. To see how this works, we consider the simplest of the exceptional LAs — $\mathfrak{g}_2$.

$\mathfrak{g}_2$

The Dynkin diagram of $\mathfrak{g}_2$ is just



$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

From exercise 8.5, we know that the roots of $\mathfrak{g}_2$, written down in terms of the Dynkin weights, take the following form:

$$\begin{matrix} & [1,1] & \\ [-1,2] & & [2,-1] \\ [-2,1] & & [1,-2] \\ & [-1,-1] & \end{matrix} \qquad \begin{matrix} [1,0] \\ [-1,1] \\ [0,-1] \end{matrix} \qquad \begin{matrix} [0,1] \\ [1,-1] \\ [-1,0] \end{matrix} .$$

We recognize these! The left-most collection are the roots of $\mathfrak{sl}_3\mathbb{C}$, while the second two columns are the weights of the fundamental and anti-fundamental of $\mathfrak{sl}_3\mathbb{C}$. So, if $\mathfrak{g}_2$ exists as a LA, we conclude that

$$\begin{aligned} \mathfrak{g}_2 \supset & \mathfrak{sl}_3\mathbb{C} \\ \mathrm{adj}(\mathfrak{g}_2) = & \mathrm{adj}(\mathfrak{sl}_3\mathbb{C}) \oplus \mathbf{3} \oplus \overline{\mathbf{3}} . \end{aligned}$$

Hence, all we need to do is to write down the 8 generators of $\mathfrak{su}(3)$ in the representation $V = \mathbf{8}\oplus\mathbf{3}\oplus\overline{\mathbf{3}}$, and then supplement them by 6 more generators corresponding to $\mathbf{3} \oplus \overline{\mathbf{3}}$. How do these act on V? It turns out pretty easy to guess, as we have natural maps $\wedge^2\mathbf{3} \to \overline{\mathbf{3}}$, and these are just right. Writing down the explicit generators and getting the normalizations straight, we can directly verify the Jacobi identity.

Exercise 12.3. Similar considerations of the weights tell us that $\Gamma_{[10]}$ of $\mathfrak{g}_2$, the 7-dimensional irrep, should split under $\mathfrak{g}_2 \supset \mathfrak{sl}_3\mathbb{C}$ as $\mathbf{7} = \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1}$. Since you are very familiar with the fundamental representation of $\mathfrak{sl}_3\mathbb{C}$, you can try to construct the $\mathbf{7}$ of $\mathfrak{g}_2$ by adding generators that transform in $\mathbf{3}$ and $\overline{\mathbf{3}}$ of $\mathfrak{sl}_3\mathbb{C}$. You should find that the Jacobi identity will fix the relative normalizations. This is an involved exercise, but I guarantee that it will teach you a lot!

So, to summarize, $\mathfrak{g}_2$ really is a LA, and we have at least learned of its basic irreps:

$$\begin{aligned} \mathfrak{g}_2 \supset \mathfrak{sl}_3\mathbb{C}\Gamma_{[10]} &= \mathbf{7} = \mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}} , \\ \Gamma_{[01]} = \mathbf{14} = \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}} . \end{aligned}$$

All irreps are real, and $\Lambda_W/\Lambda_R = 1$. Consequently, there is a unique compact Lie group G_2.

The $\mathfrak{g}_2$ has an important geometric significance for the study of manifolds. The key observation is a "simple" fact about linear algebra. Let $\varphi \in \wedge^3\mathbb{R}^7$ be a non-degenerate 3-form. $\mathbb{R}^7$ admits a natural $\mathfrak{so}(7)$ action, which leads to a corresponding action on the three forms in $\wedge^3\mathbb{R}^7$. Thus, it makes sense to consider the stabilizer sub-algebra of φ:

$$\mathrm{Stab}(\varphi) = \{x \in \mathfrak{so}(7) \ \ |x(\varphi) = 0\} \ . \tag{49}$$

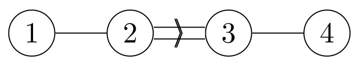
This is clearly a sub-algebra, and it turns out to be precisely $\mathfrak{g}_2$ (well, the corresponding compact form, if you like). So, another way to think of $\mathfrak{g}_2$ is via

$$\begin{aligned} \mathfrak{so}(7) &\supset \mathfrak{g}_2 \\ \mathbf{7} &= \mathbf{7} \\ \wedge^2\mathbf{7} = \mathbf{21} &= \mathbf{14} \oplus \mathbf{7} \\ \wedge^3\mathbf{7} = \mathbf{35} &= \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{27} \ , \end{aligned}$$

where $\mathbf{27} = \Gamma_{[20]}$ of $\mathfrak{g}_2$.

$\mathfrak{f}_4$

The next item in the catalog is $\mathfrak{f}_4$ with Dynkin diagram



The existence of the corresponding LA can be established in the same fashion as we argued for $\mathfrak{g}_2$. One observes

$$\mathrm{roots}(\mathfrak{f}_4) = \mathrm{roots}(\mathfrak{so}(9)) \oplus \mathrm{weights}(\mathfrak{so}(9)\ \mathrm{spinor}) \ .$$

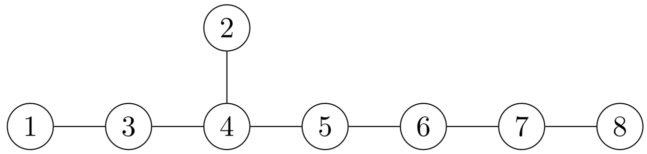
This allows one to construct the generators explicitly and check the Jacobi identity. The resulting decomposition is

$$\begin{aligned} \mathfrak{f}_4 &\supset \mathfrak{so}(9) \\ \Gamma_{[1000]} &= \mathbf{52} = \mathbf{36} \oplus \mathbf{16} \\ \Gamma_{[0001]} &= \mathbf{26} = \mathbf{1} \oplus \mathbf{9} \oplus \mathbf{16} \ . \end{aligned}$$

All other basic irreps can be obtained from tensor products of the “fundamental” **26**. The irreps are all real, and $\Lambda_W/\Lambda_R = 1$. Of all the LAs, $\mathfrak{f}_4$ is perhaps the most neglected by theoretical physicists.

$\mathfrak{e}_8$

Next we consider the E series. There are three simple LAs $\mathfrak{e}_{6,7,8}$, and like the A and D type, they have simply-laced Dynkin digrams. Let's start with the biggest of these, with Dynkin diagram



Existence of LA is established by either of two routes:

$$\begin{aligned} \text{roots}(\mathfrak{e}_8) &= \ \text{roots}(\mathfrak{su}(9)) \oplus \text{weights}(\wedge^3 \mathbf{9} \oplus \wedge^3 \overline{\mathbf{9}}) \\ &= \ \text{roots}(\mathfrak{so}(16)) \oplus \text{weights}(\mathfrak{so}(16) \ \ \text{spinor}) \ . \end{aligned}$$

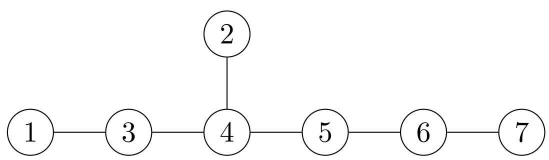
The smallest representation is the adjoint,

$$\text{adj}\, \mathfrak{e}_8 = \mathbf{248} = \Gamma_{[0,\ldots,0,1]} \ .$$

All irreps are real and $\Lambda_W/\Lambda_R = 1$. All other basic irreps can be obtained from tensor products of the adjoint.

$\mathfrak{e}_7$

The Dynkin diagram is



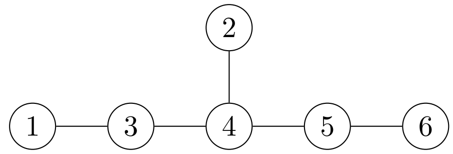
There is no question about existence, because we obtain $\mathfrak{e}_7 \oplus \mathfrak{u}(1) \subset \mathfrak{e}_8$ by striking the 8-th node in the $\mathfrak{e}_8$ diagram, as in the previous lecture. Since we already know $\mathfrak{e}_8$ exists, we obviously get generators of $\mathfrak{e}_7$. The adjoint is now $\Gamma_{[1,0,\ldots,0]}$ and has dimension 133. Unlike $\mathfrak{e}_8$ there is a representation of smaller dimension, the pseudoreal fundamental $\Gamma_{[0,\ldots,0,1]} = \mathbf{56}$. All basic irreps are obtained by tensors of $\mathbf{56}$. The reality properties are then

$$[a1, \ldots, a_7] \begin{cases} \text{is real} & \text{if} a_2 + a_5 + a_7 \ \text{ is even,} \\ \text{is pseudoreal} & \text{otherwise.} \end{cases}$$

The lattice quotient is $\Lambda_W/\Lambda_R = \mathbb{Z}_2$.

$\mathfrak{e}_6$

Finally, we come to the only exceptional LA with complex irreps and has Dynkin diagram



It can be obtained by striking the 7-th root in $\mathfrak{e}_8$ to obtain $\mathfrak{e}_8 \supset \mathfrak{e}_6 \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(2)$. There are three very key irreps:

$$\Gamma_{[100000]} = \mathbf{27} \ , \qquad \Gamma_{[000001]} = \overline{\mathbf{27}} \ , \qquad \Gamma_{[010000]} = \mathbf{78} = \text{adj} \ . \tag{50}$$

In this case $\Lambda_W/\Lambda_R = \mathbb{Z}_3$, and irreps are complex if and only if $a_1 \neq a_6$ and are real otherwise. The careful reader will have noted that all simple LAs that admit complex irreps have a $\mathbb{Z}_2$ outer automorphism obtained from the $\mathbb{Z}_2$ symmetry of the Dynkin diagram; $\mathfrak{e}_6$ is the only exceptional LA with such a structure.

13 Semisimple subalgebras of simple LAs

In this last lecture we follow Chapter XV of Cahn to give a sketch of how to find subalgebras and induced representations. The prototypical way in which subalgebras arise in physics is in the question of symmetry breaking. We consider some symmetry $\mathfrak{g}$ acting on a representation V and pick some vector $v \in V$. As long as $\mathfrak{g} \cdot v \neq 0$, this choice is not respected by the full symmetry. Instead, the remaining symmetry is a Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ defined by $\mathfrak{h} = \text{Stab}(v)$—the set of all LA elements that annihilate the vector v.

A simple example of a sub-algebra is to take $\mathfrak{h} \subset \mathfrak{sl}_2\mathbb{C}$ generated by H and X. The trouble with this choice is that it is not semisimple! In what follows, we will restrict attention to semisimple subalgebras. As many great problems, the classification of semisimple subalgebras of complex simple LAs, was solved by Dynkin. He considered two categories of subalgebras.

1. (R)egular subalgebras are $\mathfrak{g} \supset \mathfrak{h}$ where the root system of $\mathfrak{g}$ contains the root system of $\mathfrak{h}$. We already met such structures above, and in this lecture we will now describe the general story.

2. (S)pecial subalgebras are those that are not regular.

The complete classification problem can easily get out of hand even for moderately sized LAs; fortunately, we can restrict attention to maximal subalgebras. That is, given a simple LA $\mathfrak{g}$, we seek to classify $\mathfrak{g} \supset \mathfrak{h}$ such that there does not exist an $\mathfrak{h}'$ with $\mathfrak{g} \supset \mathfrak{h}' \supset \mathfrak{h}$. Non-maximal subalgebras can then be obtained recursively.

Index of embedding

When we say $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, we really mean that there is a LA homomorphism $\varphi : \mathfrak{h} \to \mathfrak{g}$, which in particular can be used to pull back the Killing form $B_\mathfrak{g}$ to $\mathfrak{h}$: $B_\mathfrak{h} = \varphi^* B_\mathfrak{g}$, or more explicitly

$B_{\mathfrak{h}}(x,y) = B_{\mathfrak{g}}(\phi(x),\phi(y))$. Now suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are both simple and V is a representation of $\mathfrak{g}$ that decomposes as $W = \bigoplus_i W_i$ under $\mathfrak{h}$. We can then use the pulled-back Killing form to relate the Dynkin indices computed with respect to $\mathfrak{h}$ and $\mathfrak{g}$. Namely, for all V we have $\ell_{\mathfrak{g}}(V) = \ell_{\varphi}\ell_{\mathfrak{h}}(W)$ for some constant ℓ_{φ} associated to the embedding. This is the index of embedding, and Dynkin proved that it is a non-negative integer. As we will see, this helps to distinguish S-subalgebras; for R-subalgebras the index of embedding is always 1.

Maximal special SSSAs

We first present Dynkin's classification of maximal special semi-simple subalgebras of a simple LA , i.e. $\mathfrak{h} \subset \mathfrak{g}$. It is divided into a few categories.

1. $\mathfrak{g}$ is a classical LA.

   (a) $\mathfrak{h}$ is non-simple and s, t are integers.

   $$\mathfrak{su}(st) \supset \mathfrak{su}(s) \oplus \mathfrak{su}(t)\ , \qquad \mathfrak{so}(st) \supset \mathfrak{so}(s) \oplus \mathfrak{so}(t)\ ,$$
   $$\mathfrak{so}(4st) \supset \mathfrak{sp}(s) \oplus \mathfrak{sp}(t)\ , \qquad \mathfrak{sp}(st) \supset \mathfrak{sp}(s) \oplus \mathfrak{so}(t)\ ,$$
   $$\mathfrak{so}(2s+2t+2) \supset \mathfrak{so}(2s+1) \oplus \mathfrak{so}(2t+1)\ .$$

   In all but the last case the fundamental decomposes into a product of the fundamentals, e.g. for $\mathfrak{su}$ case we have $\boldsymbol{st} = (\boldsymbol{s}, \boldsymbol{t})$; for the last case we have $\boldsymbol{2s+2t+2} = (\boldsymbol{2s+1}, \boldsymbol{2t+1})$. A good exercise is to describe this decomposition of fundamentals in terms of explicit matrices.

   (b) $\mathfrak{h}$ is simple with an irrep V of dimension n.

   $$\begin{array}{lll} V \ \text{real} & \Longrightarrow & \mathfrak{so}(n) \supset \mathfrak{h}\ , \\ V \ \text{pseudoreal} & \Longrightarrow & \mathfrak{sp}(n/2) \supset \mathfrak{h}\ , \\ V \ \text{complex} & \Longrightarrow & \mathfrak{su}(n) \supset \mathfrak{h}\ . \end{array}$$

   In each case the fundamental decomposes as V under $\mathfrak{h}$. For a few exceptions that occur at small n, the resulting embedding is not maximal; however, even in those cases it is a nice sub-algebra.

2. $\mathfrak{g}$ is exceptional. Dynkin gives a complete list, with $\mathfrak{h}$ labeled by index of embedding:

   $\mathfrak{g}_2 \supset \mathfrak{su}(2)^{28}$;

   $\mathfrak{f}_4 \supset \mathfrak{su}(2)^{156}$, $\mathfrak{g}_2^1 \oplus \mathfrak{su}(2)^8$;

   $\mathfrak{e}_6 \supset \mathfrak{su}(3)^9$, $\mathfrak{g}_2^3$, $\mathfrak{sp}(4)^1$, $\mathfrak{g}_2^1 \oplus \mathfrak{su}(3)^2$, $\mathfrak{f}_4^1$;

   $\mathfrak{e}_7 \supset \mathfrak{su}(2)^{399}$, $\mathfrak{su}(2)^{231}$, $\mathfrak{su}(3)^{21}$, $\mathfrak{g}_2^1 \oplus \mathfrak{sp}(3)^1$, $\mathfrak{f}_4^1 \oplus \mathfrak{su}(2)^3$, $\mathfrak{g}_2^2 \oplus \mathfrak{su}(2)^7$, $\mathfrak{su}(2)^{24} \oplus \mathfrak{su}(2)^{15}$;

   $\mathfrak{e}_8 \supset \mathfrak{su}(2)^{1240}$, $\mathfrak{su}(2)^{760}$, $\mathfrak{su}(2)^{520}$, $\mathfrak{g}_2^1 \oplus \mathfrak{f}_4^1$, $\mathfrak{su}(3)^6 \oplus \mathfrak{su}(2)^{16}$, $\mathfrak{so}(5)^{12}$.

Regular maximal SSSAs

The regular subalgebras are, in some sense, a richer class, and they can be described with the technology of extended Dynkin diagrams. We will finally understand the mysterious red nodes that were introduced on page 40 when we presented the list of Dynkin diagrams.

Extended Dynkin diagrams

Let $\mathfrak{g}$ be simple with highest root γ. We can express γ in terms of the simple roots α_i or the fundamental weights ω_i, and the two expressions are related by the Cartan matrix:

$$\gamma = \sum_i m_i \alpha_i = m_i C_{ij} \omega_j = \gamma_i \omega_i \ .$$

Thus, in particular, γ has Dynkin coefficients $\gamma_j = \sum_i m_i C_{ij}$, which can also be written very explicitly as

$$\gamma{=}2\frac{b(\gamma,\alpha_k)}{b(\alpha_k,\alpha_k)} \ .$$

The reader will note that we tabulated these Dynkin coefficients for each simple LA in the previous two lectures.

Definition 13.1. An extended root system for $\mathfrak{g}$ is obtained by adding $(-\gamma)$ to the collection of simple roots $\{\alpha_k\}$.

Exercise 13.2. Show that his can be expressed in the language of Dynkin diagrams as a simple two-step procedure:

1. add a dot to the Dynkin diagram of $\mathfrak{g}$;
2. connect the new dot with γ_j lines to the k-th node of the original diagram.

The utility of this construction is the following theorem:

Theorem 13.3. *Let $\mathfrak{h}$ be a regular maximal subalgebra of $\mathfrak{g}$. With a few exceptions, $\mathfrak{h}$ is obtained by striking a node from the extended Dynkin diagram for $\mathfrak{g}$. In all cases the result is a regular subalgebra, but it may not be maximal.*

The exceptions have been identified by Golubitsky&Rothchild. They are as follows:

$$\begin{aligned}
\mathfrak{f}_4 &\supset \mathfrak{so}(9) && \supset \mathfrak{su}(4)\oplus\mathfrak{su}(2) \ ,\\
\mathfrak{e}_7 &\supset \mathfrak{so}(12)\oplus\mathfrak{su}(2) && \supset \mathfrak{su}(4)\oplus\mathfrak{su}(4)\oplus\mathfrak{su}(2) \ ,\\
\mathfrak{e}_8 &\supset \mathfrak{so}(16) && \supset \mathfrak{so}(10)\oplus\mathfrak{su}(4) \ ,\\
\mathfrak{e}_8 &\supset \mathfrak{e}_6\oplus\mathfrak{su}(3) && \supset \mathfrak{su}(2)\oplus\mathfrak{su}(3)\oplus\mathfrak{su}(6) \ ,\\
\mathfrak{e}_8 &\supset \mathfrak{e}_7\oplus\mathfrak{su}(2) && \supset \mathfrak{su}(8)\oplus\mathfrak{su}(2) \ .
\end{aligned}$$

In each case the right-hand-side is obtained by striking a node from the extended Dynkin diagram of the left-hand-side, but the result is not maximal, as indicated.

Branching for regular subalgebras

We will now describe how we can use the roots and weights to efficiently obtain branching rules for representations of $\mathfrak{g}$ with a regular subalgebra $\mathfrak{h}$. The idea is simple. If $\lambda = \sum_i a_i \omega_i$ is the weight with respect to the original root system, with coefficients

$$a_i = \frac{2b(\lambda, \alpha_i)}{b(\alpha_i, \alpha_i)} ,$$

then in the new root system (of $\mathfrak{h}$), where we drop a root α_k and gain the root $-\gamma$, we write

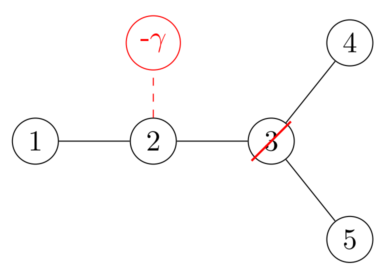
$$\lambda = \sum_{i \neq k} a_i \omega_k + a_{-\gamma} \omega_\gamma .$$

But, since we know exactly how to write γ in terms of the original roots, we can determine the last Dynkin coefficient:

$$a_{-\gamma} = -2\frac{b(\lambda, \gamma)}{b(\gamma, \gamma)} = -\sum_i a_i m_i \frac{b(\alpha_i, \alpha_i)}{b(\gamma, \gamma)} .$$

Note that the unpleasant ratio of root lengths always drops out for the ADE LAs, since all roots have the same length!

So, let's apply this to an example, where we have $\mathfrak{so}(10) \supset \mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ obtained as follows:



Exercise 13.4. We saw above that $\gamma = [0, 1, 0, 0, 0]$. Show that

$$-\gamma = -\omega_2 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) ,$$

and therefore

$$a_{-\gamma} = -(a_1 + 2a_2 + 2a_3 + a_4 + a_5) .$$

As practice, we can now decompose the weights of the fundamental representation of $\mathfrak{so}(10)$

with highest weight $\lambda = [1,0,0,0,0]$. We have the following table of weights in this representation:

weights of fundamental	$\lambda_{-\gamma}$	$[a_1, a_2, a_{-\gamma}]$	$[a_4]$	$[a_5]$
$[1,0,0,0,0]$	-1	$[1,0,-1]$	0	0
$[1,-1,0,0,0]$	1	$[1,-1,1]$	0	0
$[0,1,-1,0,0]$	0	$\underline{[0,1,0]}$	0	0
$[0,0,1,-1,-1]$	0	$[0,0,0]$	-1	-1
$[0,0,0,1,-1]$	0	$[0,0,0]$	1	-1
$[0,0,0,-1,1]$	0	$[0,0,0]$	-1	1
$[0,0,-1,1,1]$	0	$[0,0,0]$	$\underline{1}$	$\underline{1}$
$[0,-1,1,0,0]$	0	$[0,-1,0]$	0	0
$[-1,1,0,0,0]$	-1	$[-1,1,-1]$	0	0
$[-1,0,0,0,0]$	1	$[-1,0,1]$	0	0

Now this may seem a bit impenetrable at first glance, but it's really not so bad. The key is that the ordering of the weights is preserved, so that we can hunt for highest weight states of representations of $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. We indicated these by underlines. So, staring at these a bit more, we have the decomposition

$$\mathfrak{so}(10) \supset \mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$
$$\Gamma_{[1,0,0,0]} = (\Gamma_{[0,1,0]}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \Gamma_{[1]}, \Gamma_{[1]}) \ ,$$

or in perhaps more familiar language $\mathbf{10} = (\mathbf{6}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})$. This is of course a familiar story from the "usual" $\mathfrak{so}(10) \supset \mathfrak{so}(6) \oplus \mathfrak{so}(4)$ with decomposition $\mathbf{10} = (\mathbf{6}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4})$.

As another example, we can strike a node from the extended Dynkin diagram of $\mathfrak{e}_8$ to obtain $\mathfrak{e}_8 \supset \mathfrak{su}(3) \oplus \mathfrak{e}_6$. With a little work, we obtain

$$\mathbf{248} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{27}}) \ .$$

Exercise 13.5. Check the claimed decomposition by matching indices of representations.

Bon voyage!

And with this we end our little foray into representation theory. Much has been left unsaid, but I hope that what has been said will be of use in your further exploration of the subject. Examples are key! Don't be afraid to use representation theory computer packages to get intuition!

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