Zero to Two

Two-dimensional quantum field theory with (0,2) supersymmetry

Ilarion V. Melnikov
Department of Physics and Astronomy,
James Madison University,
Harrisonburg, VA 22801, USA
melnikix@jmu.edu

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Preface

> One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane.
>
> *The analytic S-matrix*
> Eden, Landshoff, Olive & Polkinghorne

Complex geometry has played a key role in most developments of the last thirty years in quantum field theory and string theory. This has come about not via the analytic S-matrix, but rather through the beautiful interrelations between supersymmetry and complex geometry. Both structures introduce a great deal of rigidity compared to the more general categories of non-supersymmetric theories and real differential geometry, and this rigidity allows for many general conceptual results and incredibly detailed quantitative predictions. Among the highlights in these developments we might recall the web of dualities between ten-dimensional string theories, the Seiberg-Witten solution of the low energy dynamics of N=2 supersymmetric gauge theories, mirror symmetry and Seiberg duality, and, more recently, the construction and investigation of a large class of non-Lagrangian field theories associated to the (2,0) superconformal theory in six dimensions and its compactifications.

On closer examination, we find that most of these remarkable results have used relations between theories with four or more supercharges and Kähler geometry. While Kähler complex manifolds are certainly the most familiar class of complex manifolds, the generic complex manifold will not carry a Kähler structure, yet it will still have a great deal more rigidity compared to a generic real manifold.

While great progress has been made in understanding theories with four or more supercharges, it is important to extend these successes as far as possible to theories with less supersymmetry. In that sense, two-dimensional quantum field theories with (0,2) supersymmetry are the "ultimate frontier" where we may still use tools from complex geometry to constrain the kinematics and dynamics of a non-trivial theory. Explorations of this frontier have been playing a growing role in modern mathematical physics.

These theories were first introduced very early on in string theory. Perturbative heterotic string theory involves the least supersymmetric theory of them all: a (0,1) two-dimensional supergravity theory, which describes the propagation of a string in a ten-dimensional background. The resulting space-time theories are chiral and propagate non-abelian gauge fields, and as a result have been studied at great length with the goal of providing unified models for elementary particle physics. In particular, if we wish to construct critical heterotic string backgrounds with minimal four-dimensional super-Poincaré invariance, then the internal degrees of freedom must be described by a (0,2) superconformal field theory. This realization led to an intensive study of such theories. A number of general techniques were constructed for obtaining a large class of models of this sort. Notably, it was recognized very early on that the connection to four-dimensional theories leads to very powerful constraints on properties of the two-dimensional theories, and insights gained from this point of view have frequently proven useful in considering more general (0,2) quantum field theories not necessarily associated with string theory.

More recently, they have come to prominence as descriptions of surface defects and low energy dynamics of solitonic strings in four-dimensional SUSY theories, where they provide some of the data that can be used to probe the dynamics of four-dimensional theories beyond perturbation theory. In addition, such theories naturally arise in the context of holography, as well as compactifications of the (2,0) six-dimensional superconformal theories on four-manifolds.

There is another conceptual reason for interest in (0,2) quantum field theories: they may be considered as models for N=1 field theories in four dimensions: some (0,2) theories exhibit confinement and supersymmetry breaking, while others have a rich IR dynamics controlled by SCFTs with chiral symmetries, marginal deformations, and accidental symmetries. So, one can certainly develop useful analogies with four-dimensional dynamics, but in a context of simpler two-dimensional theories. Indeed, the analogy can be made into a concrete relation by compactifying N=1 d=4 theories to two dimensions on an appropriate background.

The purpose of these lecture notes is to introduce the reader to these fascinating theories. The audience will be assumed to have a background in conformal theory, quantum field theory, and general relativity/differential geometry at the level of Volume I of Polchinski's string theory text. A major theme of these lecture notes will be to point out the relations between structures from complex geometry and field theory. To that end, we will need to introduce a number of mathematical concepts. Our treatment of these will not be complete, but we will strive to explain the essential results and ideas, as well as to provide references for a further study.

We will begin with a thorough examination of the basic structures of (0,2) quantum field theory and conformal field theory. While setting down the fundamentals, this will also help us to establish a set of conventions and notation that we will use in what follows. We will illustrate the structures with a few examples of exactly solvable theories. Next, we will turn to a simple class of Lagrangian theories: the (0,2) Landau-Ginzburg models and discuss the resulting renormalization group flows, dynamics, and symmetries. We will also make contact with the more familiar (2,2) theories and compare and contrast the (0,2) and (2,2) theories. Having gotten some experience with this simplest class of models, we will examine (0,2) non-linear sigma models. These theories have a very rich geometric structure and yield an important generalization of familiar Kähler geometry. They are also more delicate and exhibit anomalies that break global symmetries or even invalidate a particular theory but are particularly fascinating because of a rather direct connection with compactification of the heterotic string. After developing these structures, we will be in a position to appreciate the many simplifications offered by the (0,2) linear sigma model approach, which provides a unified framework for treating non-linear sigma models and Landau-Ginzburg theories. Here we will touch on the rich subject of mirror symmetry, mainly developed in the context of (2,2) theories and only recently generalized to classes of (0,2) models. Finally, having developed the linear sigma model technology, we will turn to a number of applications relating to IR dynamics, partition functions and elliptic genera of (0,2) theories, as well as the connection between (0,2) theories and heterotic string vacua. We will end with a description of some of the important open problems in the field.

Chapter 1

(0,2) fundamentals

Abstract

In this chapter we introduce a number of notational conventions, describe our primary object of study—the (0,2) supersymmetry algebra, and give a Lagrangian field realization of this structure.

1.1 The Lorentz group and light-cone coordinates

Consider Minkowski space $\mathbb{R}^{1,1}$ with coordinates $x^\mu = (x^0, x^1)$ and line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 \ . \tag{1.1.1}$$

We can use this coordinate basis to write 1-forms and vector fields as

$$\omega = \omega_0 dx^0 + \omega_1 dx^1 \ , \qquad\qquad v = v^0 \partial_0 + v^1 \partial_1 \ , \tag{1.1.2}$$

where we use the abbreviation $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

The proper orthochronous two-dimensional Lorentz group $\mathrm{SO}(1,1)/\mathbb{Z}_2$ is generated by boosts, and the corresponding transformations act on vector field components as

$$\Lambda(\xi) \cdot \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} \ , \tag{1.1.3}$$

where $\xi \in \mathbb{R}$ is the boost parameter.

Light-cone conventions

We will use light-cone coordinates $x^{\pm\pm} = x^0 \pm x^1$ and vector fields

$$\partial_{++} = \tfrac{1}{2}(\partial_0 + \partial_1) \ , \qquad\qquad \partial_{--} = \tfrac{1}{2}(\partial_0 - \partial_1) \ . \tag{1.1.4}$$

A vector then takes the form

$$v = v^{++}\partial_{++} + v^{--}\partial_{--} \ , \qquad v^{\pm\pm} = (v^0 \pm v^1) \ . \tag{1.1.5}$$

The dual basis to $\partial_{\pm\pm}$ is given by $dx^{\pm\pm} = (dx^0 \pm dx^1)$, and 1-forms are written as

$$\omega = \omega_{++}dx^{++} + \omega_{--}dx^{--} \ , \qquad \omega_{\pm\pm} = \tfrac{1}{2}(\omega_0 \pm \omega_1) \ . \tag{1.1.6}$$

The line element is

$$ds^2 = -dx^{++}dx^{--} \ , \qquad \Longleftrightarrow \qquad v^\mu w^\nu \eta_{\mu\nu} = -\tfrac{1}{2}(v^{++}w^{--} + v^{--}w^{++}) \ . \tag{1.1.7}$$

In other words, the components of the metric are $\eta_{++--} = \eta_{--++} = -1/2$. We use the metric with to raise and lower indices; for instance, starting with the vector v as above, we obtain a 1-form with components $v_{\mp\mp} = -\frac{1}{2}v^{\pm\pm}$.

Spinors in two dimensions

Since $\mathrm{SO}(1,1)/\mathbb{Z}_2$ is abelian, its irreducible representations are one-dimensional; this is evident for the vectors and one-forms, where the $\pm\pm$ components yield the eigenvectors:

$$\Lambda(\xi)\cdot\begin{pmatrix} v^{++} \\ v^{--}\end{pmatrix} = \begin{pmatrix} e^{\xi}v^{++} \\ e^{-\xi}v^{--}\end{pmatrix} \ , \qquad \Lambda(\xi)\cdot\begin{pmatrix}\omega_{++} & \omega_{--}\end{pmatrix} = \begin{pmatrix} e^{-\xi}\omega_{++} & e^{\xi}\omega_{--}\end{pmatrix} \ . \tag{1.1.8}$$

There are two inequivalent irreducible real spinor representations, with components denoted by $\psi_\pm$, and transformations

$$\Lambda(\xi)\cdot\psi_+ = e^{-\xi/2}\psi_+ \ , \qquad \Lambda(\xi)\cdot\psi_- = e^{\xi/2}\psi_- \ . \tag{1.1.9}$$

These are Majorana-Weyl spinors. Note that $\psi_\pm$ transform as "square roots" of the $\omega_{\pm\pm}$. We will come back to this point later, but at this point the observation is sufficient for explaining the $dx^{\pm\pm}$ notation. A little bit more generally we will say that an operator that transforms as $\Lambda(\xi)\cdot A = e^{s\xi}$, has spin s. We will only consider theories where $s \in \frac{1}{2}\mathbb{Z}$.

Exercise 1.1. The established notation allows us to write down the action for a single Majorana-Weyl fermion, an anti-commuting spinor field ψ_+:

$$S = \frac{i}{2\pi}\int d^2x \ \ \psi_+\partial_{--}\psi_+ \ .$$

Verify that this is indeed Lorentz-invariant, i.e. invariant under the transformation

$$\psi_+(x) \mapsto \Lambda(\xi)\cdot\psi_+(\Lambda^{-1}(\xi)\cdot x) = e^{-\xi/2}\psi_+(\Lambda^{-1}(\xi)\cdot x) \ .$$

Moreover, show that S is real if products of Majorana-Weyl fermions are conjugated according to

$$\overline{\chi^1\chi^2\cdots\chi^k} = \chi^k\chi^{k-1}\cdots\chi^1 \ .$$

1.2 The (0,2) supersymmetry algebra

Consider a Lorentz-invariant two-dimensional quantum field theory (QFT) defined on Minkowski space $\mathbb{R}^{1,1}$. By assumption the theory is endowed with a conserved 4-momentum operator P^μ, the components of which define the Hamiltonian $\boldsymbol{H} = P^0$, as well as the spatial momentum $\boldsymbol{P} = P^1$, and our theory possesses a (0,2) supersymmetry, if we can find a fermionic *supercharge* operator $\boldsymbol{Q}_+$ and its Hermitian conjugate $\overline{\boldsymbol{Q}}_+$ satisfying

$$\{\boldsymbol{Q}_+, \overline{\boldsymbol{Q}}_+\} = 2(\boldsymbol{H} - \boldsymbol{P}) = -4\boldsymbol{P}_{++} \ . \tag{1.2.1}$$

We will restrict attention to theories in which these conserved charges arise from conserved currents:

$$\boldsymbol{P}^\mu = \int dx^1 T^{0\mu} \ , \qquad \boldsymbol{Q}_+ = \int dx^1 S_+^0 \ , \qquad \overline{\boldsymbol{Q}}_+ = \int dx^1 \overline{S}_+^0 \ , \tag{1.2.2}$$

where $T^{\mu\nu}$ is the energy-momentum tensor, and S_+^μ and its complex conjugate $\overline{S}_+^\mu$ are supercurrents. To explore these currents further, it will be very convenient to introduce a superspace structure.

1.3 Minkowski (0,2) superspace

The commuting operators $\boldsymbol{H}$ and $\boldsymbol{P}$ generate time and spatial translations, respectively. This means, essentially by definition, that for any field ϕ we have

$$[\boldsymbol{H}, \phi] = i\partial_0\phi \ , \qquad [\boldsymbol{P}, \phi] = -i\partial_1\phi \tag{1.3.1}$$

In order to realize the action of the supercharges in terms of differential operators we introduce the (0,2) superspace, where the usual coordinates (x^0, x^1) are amended by a complex Grassmann coordinate θ^+, i.e. the full superspace has coordinates $(x^0, x^1; \theta^+, \overline{\theta}^+)$.[1] The fields are organized into superfields, and the most general superfield takes the form

$$\Phi(x;\theta) = \phi(x) + \theta^+\psi_+(x) + \overline{\theta}^+\overline{\chi}_+(x) + \theta^+\overline{\theta}^+\rho_{++}(x) \ ; \tag{1.3.2}$$

the fields ϕ and ρ_{++} have the same statistics, which are opposite to those of ψ_+ and $\overline{\chi}_+$. We call ϕ the lowest and ρ_{++} the highest component of the superfield Φ. We now realize the action of supersymmetry as follows. Let ξ^+ be an anti-commuting supersymmetry parameter. Then the supersymmetry action is generated by the Hermitian operator

$$\boldsymbol{X}_\xi = \xi^+\boldsymbol{Q}_+ - \overline{\xi}^+\overline{\boldsymbol{Q}}_+ \ , \tag{1.3.3}$$

[1] The Grassmann coordinates $\theta^+, \overline{\theta}^+$ transform under global Lorentz transformations as duals to the ψ_+ spinors introduced above. The geometry of supermanifolds governs the extension of this structure to local Lorentz invariance [1]. These ideas play an important role in covariant string perturbation theory.

i.e. the transformation of any field is given by $\delta_\xi \Phi = i[\boldsymbol{X}_\xi, \Phi]$. On the other hand, the supersymmetry action on a superfield is realized via differential operators:

$$\delta_\xi \Phi(x;\theta) = \left(\xi^+ \mathcal{Q}_+ - \overline{\xi}^+ \overline{\mathcal{Q}}_+\right) \Phi(x;\theta) \ , \tag{1.3.4}$$

where

$$\mathcal{Q}_+ = \frac{\partial}{\partial \theta^+} + 2i\overline{\theta}^+ \partial_{++} \ , \qquad \overline{\mathcal{Q}}_+ = -\frac{\partial}{\partial \overline{\theta}^+} - 2i\theta^+ \partial_{++} \tag{1.3.5}$$

realize the supersymmetry action on superspace. In particular, they satisfy

$$\mathcal{Q}_+^2 = 0 \ , \qquad \overline{\mathcal{Q}}_+^2 = 0 \ , \qquad \{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\} = -4i\partial_{++} \ . \tag{1.3.6}$$

Exercise 1.2. In this exercise we make some checks of the various definitions and gain a little practice with manipulating supercharges. First, show that our definitions are consistent with the Jacobi identity. That is, show that the following holds

$$\delta_{\xi_2}\left[\delta_{\xi_1}\Phi\right] - \delta_{\xi_1}\left[\delta_{\xi_2}\Phi\right] = [\boldsymbol{X}_{\xi_1}, [\boldsymbol{X}_{\xi_2}, \Phi]] - [\boldsymbol{X}_{\xi_2}, [\boldsymbol{X}_{\xi_1}, \Phi]] = [[\boldsymbol{X}_{\xi_1}, \boldsymbol{X}_{\xi_2}], \Phi]$$

by using the differential representation on the left and (1.2.1) on the right. Second, show that if a superfield Φ has definite statistics, i.e. $(-1)^{F_\Phi} = \pm 1$, then

$$\overline{\mathcal{Q}_+ \Phi} = (-1)^{F_\Phi} \overline{\mathcal{Q}}_+ \overline{\Phi} \tag{1.3.7}$$

if we use the Grassmann-reversing conjugation conventions as above.

1.4 Superspace derivatives and multiplets

We can construct another set of differential operators on superspace that anti-commute with $\mathcal{Q}_+$ and $\overline{\mathcal{Q}}_+$:

$$\mathcal{D}_+ = \frac{\partial}{\partial \theta^+} - 2i\overline{\theta}^+ \partial_{++} \ , \qquad \overline{\mathcal{D}}_+ = -\frac{\partial}{\partial \overline{\theta}^+} + 2i\theta^+ \partial_{++} \ . \tag{1.4.1}$$

We find $\mathcal{D}_+^2 = \overline{\mathcal{D}}_+^2 = 0$ and $\{\mathcal{D}_+, \overline{\mathcal{D}}_+\} = 4i\partial_{++}$. These supercovariant derivatives allow us to introduce a wide class of constrained superfields and hence supersymmetry multiplets. Of particular importance are *chiral* superfields, which satisfy

$$\overline{\mathcal{D}}_+ \Phi = 0 \qquad \Longleftrightarrow \qquad \Phi = \phi - 2i\theta^+ \psi_+ - 2i\theta^+ \overline{\theta}^+ \partial_{++}\phi \ , \tag{1.4.2}$$

as well as anti-chiral superfields

$$\mathcal{D}_+ \overline{\Phi} = 0 \qquad \Longleftrightarrow \qquad \overline{\Phi} = \overline{\phi} - 2i\overline{\theta}^+ \overline{\psi}_+ + 2i\theta^+ \overline{\theta}^+ \partial_{++}\overline{\phi} \ . \tag{1.4.3}$$

Here the ψ_+ and $\overline{\psi}_+$ are components of a right-moving Weyl fermion, and the bar reflects the naive conjugation. That is, if we write ψ_+ in terms of Majorana-Weyl components as $\psi_+ = \psi_+^1 + i\psi_+^2$, then $\overline{\psi}_+ = \psi_+^1 - i\psi_+^2$.

Exercise 1.3. Derive the supersymmetry transformations for the components of chiral and anti-chiral superfields:

$$\delta_\xi \phi = -2i\xi^+\psi_+ \ , \qquad \delta_\xi \psi_+ = 2\overline{\xi}^+\partial_{++}\phi \ ; \qquad \delta_\xi \overline{\phi} = -2i\overline{\xi}^+\overline{\psi}_+ \ , \qquad \delta_\xi \overline{\psi}_+ = 2\xi^+\partial_{++}\overline{\phi} \ .$$

We can generalize the chirality condition by demanding that the $\overline{\mathcal{D}}_+$ derivative of a superfield is some fixed chiral superfield E. Such *E-chiral* superfields will play an important role in describing the left-moving degrees of freedom. Another important class of superfields is obtained by imposing a reality condition $V = \overline{V}$.

A superfield is *irreducible* if and only if it does not contain a proper sub-multiplet that is closed under supersymmetry. A reducible superfield is *decomposable* if and only if it can be written as a sum of two non-trivial irreducible superfields. Thus, a general chiral superfield is irreducible; an E-chiral superfield is reducible (because E_+ is a non-trivial sub-multiplet) but indecomposable (because its complement is not a closed sub-multiplet); a superfield given by a sum a chiral superfield and an anti-chiral superfield is reducible and decomposable.

All of the superfields we discuss will have definite statistics, and we will distinguish these as *even* and *odd*, depending on whether the lowest component is a boson or fermion.

There is more that may be said about the structure of superspace for two-dimensional theories; we refer the reader to [2] for a wealth of details and background. The (0,2) superspace was introduced in [3].

1.5 Supersymmetric actions and fermi multiplets

The superspace formalism is particularly suited to constructing supersymmetric Lagrangians. As in four dimensions, the basic observations are that the top component of any superfield transforms as a total derivative, as does the middle component of a chiral or an antichiral superfield. Hence, up to boundary terms a real supersymmetric action can be written as

$$S = \frac{1}{\pi}\int d^2x \ \left\{ \mathcal{D}_+\overline{\mathcal{D}}_+ \ K_{--} - \frac{i}{2}\mathcal{D}_+ W_- - \frac{i}{2}\overline{\mathcal{D}}_+\overline{W}_- \right\} \ , \qquad (1.5.1)$$

where K_{--} is an even real superfield, W_- is an odd chiral superfield, and $\overline{W}_-$ is its conjugate.

The first term is a D-term and will contain kinetic terms and their supersymmetric completion, while the last two terms are F-term superpotential terms; they will enjoy the usual non-renormalization theorems that follow from holomorphy, symmetries, and selection rules.

In order to write a supersymmetric potential interaction we need to introduce left-moving fermions. The simplest way to do this is with a *chiral fermi* superfield, which takes the form

$$\Gamma_- = \gamma_- - 2i\theta^+ G_{-+} - 2i\theta^+\overline{\theta}^+\partial_{++}\gamma_- \qquad (1.5.2)$$

and has a conjugate antichiral superfield

$$\overline{\Gamma}_- = \overline{\gamma}_- + 2i\overline{\theta}^+\overline{G}_{-+} + 2i\theta^+\overline{\theta}^+\partial_{++}\overline{\gamma}_- \ . \qquad (1.5.3)$$

We can now write a superpotential of the form $W_- = \Gamma_- J(\Phi)$. However, we can be more general by taking the Γ_- to be E-chiral fermi superfields satisfying

$$\overline{\mathcal{D}}_+\Gamma_- = -2iE(\Phi) \ . \tag{1.5.4}$$

This leads to the component expansion

$$\begin{aligned}\Gamma_- &= \gamma_- - 2i\theta^+ G_{-+} - 2i\theta^+\overline{\theta}^+\partial_{++}\gamma_- + 2i\overline{\theta}^+ E(\Phi) \ ,\\ \overline{\Gamma}_- &= \overline{\gamma}_- + 2i\overline{\theta}^+\overline{G}_{-+} + 2i\theta^+\overline{\theta}^+\partial_{++}\gamma_- - 2i\theta^+\overline{E}(\overline{\Phi}) \ .\end{aligned} \tag{1.5.5}$$

Exercise 1.4. Compute the supersymmetry variations for the fermi multiplets:

$$\begin{aligned}\delta_\xi\gamma_- &= -2i\xi^+ G_{-+} + 2i\overline{\xi}^+ E(\phi) \ , &\qquad \delta_\xi\overline{\gamma}_- &= 2i\overline{\xi}^+\overline{G}_{-+} - 2i\xi^+\overline{E} \ ,\\ \delta_\xi G_{-+} &= -2i\overline{\xi}^+\partial_{++}\gamma_- + 2\overline{\xi}^+ E'(\phi)\psi_+ \ , &\qquad \delta_\xi\overline{G}_{-+} &= -2i\xi^+\partial_{++}\overline{\gamma}_- - 2i\xi^+\overline{E}'(\overline{\phi})\overline{\psi}_+ \ .\end{aligned}$$

Suppose we have M E-chiral multiplets Γ_-^A satisfying $\overline{\mathcal{D}}_+\Gamma_-^A = -2iE^A(\Phi)$, and we set the superpotential to be

$$W_- = \sum_{A=1}^{M}\Gamma_-^A J_A(\Phi) \ . \tag{1.5.6}$$

The action (1.5.1) will be supersymmetric if and only if W_i is an off-shell chiral superfield; equivalently, the E and J satisfy the supersymmetry constraint

$$\sum_{A=1}^{M} E^A(\Phi)J_A(\Phi) = 0 \ . \tag{1.5.7}$$

We will refer to this as the $E \cdot J$ *constraint* in what follows. As we will see, there are many interesting solutions to this constraint, but of course a particularly simple one is $E^A = 0$ *or* $J_A = 0$ for all A.

1.6 (0,2) Yukawa models

We now have all the ingredients to write down the most general classical 2-derivative $(0,2)$-supersymmetric action where all fields have spin $s \leq 1/2$. In this case the bosons and their superpartners are arranged in even chiral multiplets, while the left-moving fermions are lowest components of E-chiral fermi multiplets just discussed.

A particularly important class of theories are those with free kinetic terms for bose chiral multiplets, E-chiral fermi multiplets and their conjugates, which we will call the *(0,2) Yukawa models*. The kinetic terms are quadratic in the fields and take the form

$$K_{--} = -\frac{i}{8}\left[\,\overline{\Phi}\partial_{--}\Phi - \Phi\partial_{--}\overline{\Phi}\right] - \frac{1}{4}\overline{\Gamma}_-\Gamma_- \ . \tag{1.6.1}$$

Exercise 1.5. Compute the component expansion and verify that it leads to the Lagrangian

$$\pi\mathcal{L} = \frac{1}{2}(\partial_{++}\overline{\phi}\partial_{--}\phi + \partial_{++}\phi\partial_{--}\overline{\phi}) + i\overline{\psi}_{+}\partial_{--}\psi_{+} + i\overline{\gamma}_{-}\partial_{++}\gamma_{-} + \overline{G}_{-+}G_{-+} + \overline{\psi}_{+}\overline{E}'\gamma_{-} + \overline{\gamma}_{-}E'\psi_{+} - \overline{E}E\ ,$$

where $E'(\phi) = \partial E/\partial\phi$.

As we see from the exercise, when we set $E = 0$ we get a free massless action. More generally, we obtain an interacting theory, and we can spruce up the interactions further by introducing the potential J as above. A short computation shows that this leads to additional terms in the action

$$\pi\mathcal{L}_{\mathrm{J}} = -G_{-+}J(\phi) - \overline{G}_{-+}\overline{J}(\phi) + \gamma_{-}J'(\phi)\psi_{+} + \overline{\psi}_{+}\overline{J}'(\overline{\phi})\overline{\gamma}_{-}\ . \tag{1.6.2}$$

Hence, after eliminating the auxiliary G_{-+} fields by their equations of motion, we obtain a sum of three terms:

$$\begin{aligned}\pi\mathcal{L}_{\mathrm{kin}} &= \frac{1}{2}(\partial_{++}\overline{\phi}\partial_{--}\phi + \partial_{++}\phi\partial_{--}\overline{\phi}) + i\overline{\psi}_{+}\partial_{--}\psi_{+} + i\overline{\gamma}_{-}\partial_{++}\gamma_{-}\ ,\\ \pi\mathcal{L}_{\mathrm{E}} &= +\overline{\gamma}_{-}E'\psi_{+} + \overline{\psi}_{+}\overline{E}'\gamma_{-} - \overline{E}E\ ,\\ \pi\mathcal{L}_{\mathrm{J}} &= +\gamma_{-}J'\psi_{+} + \overline{\psi}_{+}\overline{J}'\overline{\gamma}_{-} - \overline{J}J\ .\end{aligned} \tag{1.6.3}$$

We need to solve the $E\cdot J$ constraint in order for the action to be supersymmetric. With just one fermi multiplet our choices are meager: we must set either $E = 0$ or $J = 0$. The two classes of theories so obtained are equivalent by the simple relabeling $\gamma_{-}\to\overline{\gamma}_{-}$, $\overline{\gamma}_{-}\to\gamma_{-}$.

We have presented the action for just one bose and one fermi multiplet, but it is easy to generalize it to many multiplets. The E and J couplings now carry indices as above, and field redefinitions allow us to bring the quadratic kinetic term to a canonical form (written with the summation convention for repeated indices)

$$K_{--} = -\frac{i}{8}\delta_{a\overline{b}}\left[\,\overline{\Phi}^{\overline{b}}\partial_{--}\Phi^{a} - \Phi^{a}\partial_{--}\overline{\Phi}^{\overline{b}}\right] - \frac{1}{4}\delta_{A\overline{B}}\overline{\Gamma}_{-}^{\overline{B}}\Gamma_{-}^{A}\ , \tag{1.6.4}$$

and it is easy to add the additional index structure to the component action; for example, we have $\gamma_{-}J'\psi_{+} = \sum_{A,b}\gamma_{-}^{A}\frac{\partial J}{\partial\phi^{a}}\psi_{+}^{a}$. We will keep the indices as light as possible and only write out expressions in full glory when absolutely necessary.

Exercise 1.6. At first sight we might be tempted to consider an exciting generalization for the quadratic action:

$$\Delta K_{--} = \frac{1}{8}\tau_{AB}\Gamma_{-}^{A}\Gamma_{-}^{B} - \frac{1}{8}\tau_{AB}^{*}\overline{\Gamma}_{-}^{A}\overline{\Gamma}_{-}^{B}$$

for some constant anti-symmetric matrix τ_{AB}. Show that this is a mirage: this coupling can be absorbed into a shift $J_{B}^{\mathrm{new}} = J_{B}^{\mathrm{old}} + E^{A}\tau_{AB}$; note that the shift is consistent with the $E\cdot J$ constraint.

At this point we actually have the basic structure of a renormalizable Lagrangian (0,2) field theory. There are two important generalizations that we will discuss in detail below:

1. we will consider a non-linear kinetic term and write a (0,2) non-linear sigma model — this will lead to many important connections with heterotic geometry;

2. we can also introduce vector fields by gauging some global symmetry of our theory — this is the domain of (0,2) gauged linear sigma models.

One can also discuss the most general gauging of a symmetry on some curved target-space geometry, but that is usually too much of a good thing, and we will stick to these simpler generalizations.

Superspace equations of motion

In this section we will illustrate a convenient superspace trick for deriving the equations of motion. The trick relies on three points:

1. if $\mathcal{D}_+\overline{\mathcal{D}}_+ [AX]_{\theta^+,\overline{\theta}^+=0} = 0$ for all (0,2) superfields X, then $A = 0$;

2. the variation of a bose anti-chiral field may be written as $\delta\overline{\Phi} = \mathcal{D}_+\mathcal{X}_-$ for some general odd (0,2) superfield $\mathcal{X}_-$;

3. the variation of a fermi E-anti-chiral field may be written as $\delta\overline{\Gamma}_- = \mathcal{D}_+Y_{--} - 2i\overline{E}'\mathcal{X}_-$, where the second term imposes the constraint $\mathcal{D}_+\delta\overline{\Gamma}_- = -2i\delta\overline{E}(\overline{\Phi})$.

Plugging these variations into the action and using the $E \cdot J$ constraint, we find the following equations of motion for the (0,2) theory with K_{--} as in (1.6.4) and potential W_- as in (1.5.6).

$$\mathcal{D}_+\Gamma_- + 2i\overline{J} = 0 \ , \qquad \partial_{--}\mathcal{D}_+\Phi - 2\Gamma_-\overline{E}' - 2\overline{\Gamma}_-\overline{J}' = 0 \ . \tag{1.6.5}$$

Taking the lowest components of these, we find

$$G_{-+} - \overline{J} = 0 \ , \qquad i\partial_{--}\psi_+ + \overline{J}'\overline{\gamma}_- + \overline{E}'\gamma_- = 0 \ , \tag{1.6.6}$$

and this indeed matches the equations of motion from the component action.

Another perspective on the E-couplings

%%% insert Jacques' fermionic gauge symmetry picture.

1.7 The supercurrent algebra via superspace

One of the most important and yet accessible structures in a QFT is the current algebra — the equal time commutation relations of conserved currents associated to continuous global symmetries. In the context of (0,2) QFT we are then led to some natural questions: what is the structure of the (0,2) supercurrent algebra? what superspace multiplets encode this structure? Although aspects of this structure were discussed long ago [4], a complete study was carried out much more recently in [5], and we will now summarize the assumptions and results of that work.

Assumptions on the current algebra

The main result of [5] is that the supercurrent algebra and the supercurrent multiplet are fixed by several basic assumptions:

1. the multiplet includes the real, conserved, and symmetric energy-momentum tensor $T_{\mu\nu}$;
2. the multiplet includes the supercurrents $S_{+\mu}$ and $\overline{S}_{+\mu}$;
3. the multiplet does not have any operators with absolute value of spin greater than 2;
4. the multiplet is not decomposable;
5. the multiplet is consistent with (1.2.1).

This is worked out in detail for theories with ≤ 4 supercharges and dimensions, but we will just present the results for two-dimensional (0,2) theories.

While the first two assumptions are perhaps "obvious," it is the third that is most constraining on the form of the relevant superfields. To start, consider the energy-momentum tensor

$$T_{\mu\nu}dx^\mu dx^\nu = T_{++++}dx^{++}dx^{++} + 2T_{++--}dx^{++}dx^{--} + T_{----}dx^{--}dx^{--} \ , \qquad (1.7.1)$$

with

$$T_{++--} = T_{--++} \ , \qquad \partial_{++}T_{\pm\pm--} + \partial_{--}T_{\pm\pm++} = 0 \ . \qquad (1.7.2)$$

We immediately see that T_{----} and T_{++++} must appear as, respectively, the lowest and highest components of two distinct even superfields. Furthermore, the lowest component of the superfield containing T_{++++} must be a spin -1 object j_{++}.

Since

$$\boldsymbol{P}_{++} = -\int dx^1\,(T_{++++} + T_{++--}) \ , \qquad \boldsymbol{Q}_{+} = -\int dx^1\,(S_{+++} + S_{+--}) \ , \qquad (1.7.3)$$

in order to obtain (1.2.1), we require that the current algebra has the terms

$$\{\overline{\boldsymbol{Q}}_+, S_{+++}\} = -4T_{++++} + \dots \ , \qquad \{\overline{\boldsymbol{Q}}_+, S_{+--}\} = -4T_{++--} + \dots \ , \tag{1.7.4}$$

where the ... refer to Schwinger terms that integrate to zero without additional operator insertions. Demanding that these relations arise from the superfields as above leads to an intricate set of algebraic conditions. The result of [5] is that these conditions completely determine the general supercurrent multiplet, and we present their results next.

The $\mathcal{S}$ and $\mathcal{R}$ multiplets

The most general supercurrent multiplet, *the $\mathcal{S}$-multiplet*, consists of two real even superfields $\mathcal{S}_{++}$ and $\mathcal{T}_{----}$, as well as a complex odd superfield $\mathcal{W}_-$ subject to relations [2]

$$4\partial_{--}\mathcal{S}_{++} = \mathcal{D}_+\mathcal{W}_- + \overline{\mathcal{D}_+\mathcal{W}_-} \ , \qquad \overline{\mathcal{D}}_+\mathcal{T}_{----} = \partial_{--}\mathcal{W}_- \ , \qquad \overline{\mathcal{D}}_+\mathcal{W}_- = C \ . \tag{1.7.5}$$

These have the component expansions

$$\begin{aligned}
\mathcal{S}_{++} &= j_{++} - i\theta^+ S_{+++} - i\overline{\theta}^+\overline{S}_{+++} - 4\theta^+\overline{\theta}^+ T_{++++} \ , \\
\mathcal{W}_- &= -\overline{S}_{+--} - i\theta^+(4T_{++--} + 2i\partial_{--}j_{++}) - \overline{\theta}^+ C + 2i\theta^+\overline{\theta}^+\overline{S}_{+--,++} \ , \\
\mathcal{T}_{----} &= 2T_{----} - \theta^+\partial_{--}S_{+--} + \overline{\theta}^+\partial_{--}\overline{S}_{+--} + 2\theta^+\overline{\theta}^+\partial_{--}^2 j_{++} \ ,
\end{aligned} \tag{1.7.6}$$

Exercise 1.7. Show that the relations imply that the energy-momentum tensor and supercurrents are conserved, while C is a constant.

Exercise 1.8. Check that the $\mathcal{S}$-multiplet implies the following supercurrent algebra:

$$\begin{aligned}
&\{\boldsymbol{Q}_+, \overline{S}_{+++}\} = -4T_{++++} + 2i\partial_{++}j_{++} \ , \qquad &&\{\boldsymbol{Q}_+, \overline{S}_{+--}\} = -4T_{++--} - 2i\partial_{--}j_{++} \ , \\
&\{\boldsymbol{Q}_+, S_{+++}\} = 0 \ , &&\{\boldsymbol{Q}_+, S_{+--}\} = i\overline{C} \ .
\end{aligned} \tag{1.7.7}$$

Note that if and only if $C = 0$ we can integrate the $\mathcal{S}$-supercurrent algebra to

$$\boldsymbol{Q}_+^2 = \overline{\boldsymbol{Q}}_+^2 = 0 \ , \qquad \{\boldsymbol{Q}_+, \overline{\boldsymbol{Q}}_+\} = -4P_{++} \ . \tag{1.7.8}$$

While any (0,2) QFT must admit an $\mathcal{S}$-multiplet, some admit a more restricted structure. An important simplification occurs when $\mathcal{W}_- = i\overline{\mathcal{D}}_+\mathcal{R}_{--}$ for some real operator $\mathcal{R}_{--}$. This requires $C = 0$ and, after relabeling $\mathcal{S}_{++} \to \mathcal{R}_{++}$, yields the *$\mathcal{R}$-multiplet*:

$$\partial_{--}\mathcal{R}_{++} + \partial_{++}\mathcal{R}_{--} = 0 \ , \qquad \overline{\mathcal{D}}_+(\mathcal{T}_{----} - i\partial_{--}\mathcal{R}_{--}) = 0 \ , \tag{1.7.9}$$

with component expansions

$$\mathcal{R}_{\pm\pm} = j_{\pm\pm} - i\theta^+ S_{+\pm\pm} - i\overline{\theta}^+\overline{S}_{+\pm\pm} - 4\theta^+\overline{\theta}^+ T_{++\pm\pm} \ . \tag{1.7.10}$$

[2] We have slightly different conventions for the light-cone and conversions of bispinors from [5] that account for the different factors here and in the algebra below.

An $\mathcal{R}$-multiplet requires that the theory has a conserved R-current with components $j_{\pm\pm}$. The resulting supercurrent algebra is

$$\begin{aligned} \{\boldsymbol{Q}_+, \overline{S}_{+\pm\pm}\} &= -4T_{++\pm\pm} + 2i\partial_{++}j_{\pm\pm} \ , & \{\boldsymbol{Q}_+, S_{+\pm\pm}\} &= 0 \ , \\ [\boldsymbol{Q}_+, j_{\pm\pm}] &= S_{+\pm\pm} \ , & [\overline{\boldsymbol{Q}}_+, j_{\pm\pm}] &= -\overline{S}_{+\pm\pm} \ . \end{aligned} \tag{1.7.11}$$

Defining the conserved R-charge as

$$\boldsymbol{R} = -\int dx^1 \, (j_{++} + j_{--}) \ , \tag{1.7.12}$$

we see that we can integrate the $\mathcal{R}$-supercurrent algebra to

$$\{\boldsymbol{Q}_+, \overline{\boldsymbol{Q}}_+\} = -4P_{++} \ , \qquad [\boldsymbol{R}, \boldsymbol{Q}_+] = -\boldsymbol{Q}_+ \ , \qquad [\boldsymbol{R}, \overline{\boldsymbol{Q}}_+] = +\overline{\boldsymbol{Q}}_+ \ . \tag{1.7.13}$$

An $\mathcal{R}$ multiplet exists in most of the (0,2) theories we will discuss in detail in this review.

A superconformal theory

A larger symmetry structure emerges when $\mathcal{W}_- = \partial_{--}\overline{\mathcal{D}}_+U$ for some real operator U.

Exercise 1.9. Show that given a solution to (1.7.5) and a real operator U there exists an improvement

$$\mathcal{W}_-^{\text{new}} = \mathcal{W}_- - \partial_{--}\overline{\mathcal{D}}_+U \ , \quad \mathcal{S}_{++}^{\text{new}} = \mathcal{S}_{++} - \tfrac{1}{4}[\mathcal{D}_+, \overline{\mathcal{D}}_+]U \ , \quad \mathcal{T}_{----}^{\text{new}} = \mathcal{T}_{----} - \partial_{--}^2 U$$

such that the "new" operators also satisfy (1.7.5). Thus, whenever $\mathcal{W}_- = -\partial_{--}\overline{\mathcal{D}}_+U$, we can obtain a new multiplet with $\mathcal{W}_-^{\text{new}} = 0$.

When we are able to set $\mathcal{W}_- = 0$ by an improvement transformation the theory is in fact superconformal. That is, $T_{++--} = 0$, so that the energy-momentum tensor is traceless, T_{----} is invariant under (0,2) supersymmetry, and $\mathcal{S}_{++}$ is a right-moving multiplet: $\partial_{--}\mathcal{S}_{++} = 0$.

A few comments on the supermultiplets

The structure of the supercurrent multiplets can be used to make many exact statements about renormalization group (RG) flow of a theory. A number of these are discussed in [5], and it is pointed out that one must keep a few basic facts in mind. Consider an RG flow obtained by deforming a conformal theory by a relevant operator. In this case, the extreme UV theory has a superconformal multiplet, and the relevant deformation causes it to mix with other operators, leading to a larger multiplet. However, once we determine the multiplet structure at a large but finite cut-off, it must persist at all energy scales (of course it is possible that in the extreme IR we find another superconformal multiplet).

The form of the supercurrent algebra should not to be confused with properties of the vacuum. Consider, for instance, a theory with a UV supercurrent $\mathcal{S}$-multiplet with the constant $C = 0$. In this case it is sensible to ask whether (0,2) supersymmetry is spontaneously broken by the vacuum, i.e. whether the ground state is annihilated by $\boldsymbol{P}_{++}$. As pointed out in [6] (in a more general context), the supersymmetry algebra implies that either all or none of the supercharges will be spontaneously broken.

On the other hand, if we find that the UV theory has an $\mathcal{S}$-multiplet with $C \neq 0$, then we say that the supercurrent algebra is *deformed* from its standard form. It is a simple exercise to show that if we set

$$\Sigma_{+\pm\pm} = e^{i\alpha} S_{+\pm\pm} + e^{-i\alpha} \overline{S}_{+\pm\pm} \ , \tag{1.7.14}$$

and denote by $\boldsymbol{Q}'_+ = e^{i\alpha}\boldsymbol{Q}_+ + e^{-i\alpha}\overline{\boldsymbol{Q}}_+$, then we can choose a phase α so that the real supercurrent Σ belongs to a standard (0,1) current algebra

$$\{\boldsymbol{Q}'_+, \Sigma_{+\pm\pm}\} = -4T_{++\pm\pm} \ . \tag{1.7.15}$$

It is then sensible to ask whether $\boldsymbol{Q}'_+$ is spontaneously broken — a difficult question since we no longer have the nice tools of holomorphy at our disposal.[3]

The supercurrent algebra of the Yukawa models

The Yukawa model constructed above provides a simple realization of these multiplets. Let

$$\begin{aligned}
\mathcal{S}_{++} &= \frac{1}{4}\overline{\mathcal{D}}_+\overline{\Phi}\mathcal{D}_+\Phi \ , \\
\mathcal{W}_- &= -2(E\overline{\Gamma}_- + \Gamma_- J) \ , \\
\mathcal{T}_{----} &= -2\partial_{--}\Phi\partial_{--}\overline{\Phi} - i\left(\overline{\Gamma}_-\partial_{--}\Gamma_- + \Gamma_-\partial_{--}\overline{\Gamma}_-\right) \ .
\end{aligned} \tag{1.7.16}$$

Exercise 1.10. Use the $E \cdot J$ constraint and the superspace equations of motion from (1.6.5) to show that these components satisfy the $\mathcal{S}$-multiplet relations of (1.7.5).

If we set $E = 0$, then up to the equations of motion we have

$$\mathcal{W}_- = i\overline{\mathcal{D}}_+\mathcal{R}_{--} \ , \qquad\qquad \mathcal{R}_{--} = \Gamma_-\overline{\Gamma}_- \ . \tag{1.7.17}$$

This is quite sensible since in this case the theory has an R-symmetry that assigns charge $+1$ to θ^+ and Γ^- and leaves the even chiral superfields invariant. This gives us the structure of a $\mathcal{R}$-multiplet.

Finally, if we set $E = J = 0$, we obtain a free superconformal field theory with a superconformal multiplet.

[3]The case when $\boldsymbol{Q}'_+$ is not spontaneously broken is what is sometimes referred to as "partial supersymmetry breaking." We will not use this terminology.

1.8 Euclidean worldsheet

We will most often find it convenient to work on a Euclidean world-sheet Σ ; typically this will be $\mathbb{R}^2 \simeq \mathbb{C}$, but for some applications we will find it convenient to compactify this to a sphere $S^2 \simeq \mathbb{P}^1$, and sometimes we will discuss the theory on the cylinder $\mathbb{R} \times S^1$ or the torus T^2.

Starting with the Minkowski theory, we define the Euclidean theory on $\mathbb{R}^2$ with coordinates (y^1, y^2) by analytic continuation $x^0 = -iy^1$ and a convenient relabeling $y^2 = -x^1$. We define a holomorphic coordinate $z = y^1 + iy^2$, so that the analytic continuation leads to

$$x^{++} = -iy^1 - y^2 = -i\overline{z} , \qquad x^{--} = -iy^1 + y^2 = -iz ,$$
$$\partial_{++} = i\frac{\partial}{\partial \overline{z}} , \qquad \partial_{--} = i\frac{\partial}{\partial z} . \tag{1.8.1}$$

We will often abbreviate the derivatives $\partial/\partial\overline{z}$ and $\partial/\partial z$ as $\bar{\partial}$ and ∂. We will also use, as in [7], the conventions that the integration measure on $\mathbb{R}^2$ is $d^2z = idz \wedge d\overline{z} = 2dx^1dx^2$, while the Dirac delta function $\delta^2(z,\overline{z}) = \frac{1}{2}\delta(x^1)\delta(x^2)$. This implies the very useful identity

$$\partial_z \, \overline{z}^{-1} = 2\pi\delta^2(z,\overline{z}) . \tag{1.8.2}$$

With these conventions, given a theory defined a Minkowski action $S_M = \int dx^0dx^1\mathcal{L}_M$, the corresponding Euclidean action $S_E = \int dy^1dy^2\mathcal{L}_E$ is obtained via

$$\mathcal{L}_E = -\mathcal{L}_M[\partial_{++} = i\bar{\partial}; \partial_{--} = i\partial] . \tag{1.8.3}$$

Euclidean fermions

The Euclidean Lorentz group is SO(2) = U(1), and its action on various representations is obtained by replacing $\xi \to i\xi$ in (1.1.8) and similar expressions above. In particular, a 1-form $\omega = \omega_z dz + \omega_{\overline{z}} d\overline{z}$ has components that transform as

$$\Lambda(\xi) \cdot \begin{pmatrix} \omega_{\overline{z}} & \omega_z \end{pmatrix} = \begin{pmatrix} e^{-i\xi}\omega_{\overline{z}} & e^{i\xi}\omega_z \end{pmatrix} . \tag{1.8.4}$$

Similarly, we see that irreducible spinor representations are the two 1-dimensional Weyl representations,

$$\Lambda(\xi) \cdot \psi_+ = e^{-i\xi/2}\psi_+ , \qquad \Lambda(\xi) \cdot \psi_- = e^{i\xi/2}\psi_- . \tag{1.8.5}$$

Clearly, unlike in two-dimensional Minkowski space, it is not possible to define a Majorana-Weyl spinor. This sometimes leads to some unnecessary confusion, so let us be quite clear: it is perfectly sensible to discuss the Euclidean version of a Minkowski theory of a Majorana-Weyl fermion. Starting with the action in Exercise 1.1, and performing the continuation we obtain the Euclidean action

$$S_E = \frac{1}{2\pi} \int d^2z \; \psi_+ \partial_z \psi_+ . \tag{1.8.6}$$

S_E does not have any obvious reality properties, but this is no cause for alarm, precisely because it is a Euclidean continuation of a unitary Minkowski theory. A related point is that when we continue a Weyl fermion with components $(\psi, \overline{\psi})$, then in the Euclidean theory we must treat the two components as independent degrees of freedom.

Superspace

Finally, we give our conventions for Euclidean superspace. Now θ^+ and $\overline{\theta}^+$ should be treated as independent Grassmann variables, and we define

$$\theta^+ = \frac{i}{\sqrt{2}}\theta \ , \qquad \overline{\theta}^+ = -\frac{i}{\sqrt{2}}\overline{\theta} \ , \tag{1.8.7}$$

so that

$$\mathcal{D}_+ = -i\sqrt{2}\mathcal{D} \ , \qquad \overline{\mathcal{D}}_+ = -i\sqrt{2}\overline{\mathcal{D}} \ , \qquad \mathcal{Q}_+ = -i\sqrt{2}\mathcal{Q} \ , \qquad \overline{\mathcal{Q}}_+ = -i\sqrt{2}\overline{\mathcal{Q}} \tag{1.8.8}$$

with

$$\begin{aligned} \mathcal{D} &= \partial_\theta + \overline{\theta}\bar{\partial} \ , & \mathcal{Q} &= \partial_\theta - \overline{\theta}\bar{\partial} \ , \\ \overline{\mathcal{D}} &= \partial_{\overline{\theta}} + \theta\bar{\partial} \ , & \overline{\mathcal{Q}} &= \partial_{\overline{\theta}} - \theta\bar{\partial} \ . \end{aligned} \tag{1.8.9}$$

The non-trivial anti-commutators for these are $\{\mathcal{D}, \overline{\mathcal{D}}\} = 2\bar{\partial}$ and $\{\mathcal{Q}, \overline{\mathcal{Q}}\} = -2\bar{\partial}$.

Moving on to the superfields, we will make the $\theta, \overline{\theta}$ substitutions as above, and we will also drop the $\pm$ on the fermions to stream-line notation. This leads to the Euclidean superfields

$$\begin{aligned} \Phi &= \phi + \sqrt{2}\theta\psi + \theta\overline{\theta}\bar{\partial}\phi \ , & \Gamma &= \gamma + \sqrt{2}\theta G + \theta\overline{\theta}\bar{\partial}\gamma + \sqrt{2}\overline{\theta}E(\Phi) \ , \\ \overline{\Phi} &= \overline{\phi} - \sqrt{2}\overline{\theta}\overline{\psi} - \theta\overline{\theta}\bar{\partial}\overline{\phi} \ , & \overline{\Gamma} &= \overline{\gamma} + \sqrt{2}\overline{\theta}\overline{G} - \theta\overline{\theta}\bar{\partial}\overline{\gamma} + \sqrt{2}\theta\overline{E}(\overline{\Phi}) \ . \end{aligned} \tag{1.8.10}$$

These satisfy $\overline{\mathcal{D}}\Phi = 0$ and $\overline{\mathcal{D}}\Gamma = \sqrt{2}E$, as well as $\mathcal{D}\overline{\Phi} = 0$ and $\mathcal{D}\overline{\Gamma} = \sqrt{2}\overline{E}$. As we mentioned above, the Euclidean fermions ψ and $\overline{\psi}$, as well as γ and $\overline{\gamma}$, are now to be treated as independent variables.[4] However, the continued theory still has a charge conjugation operator $\mathcal{C}$ which acts as follows: it conjugates the Grassmann–even fields, reverses the order of the Grassmann–odd fields, and sends

$$\mathcal{C}: \quad \theta \mapsto -i\overline{\theta} \ , \quad \overline{\theta} \mapsto -i\theta \ , \quad \psi \mapsto +i\overline{\psi} \ , \quad \overline{\psi} \mapsto +i\psi \ , \quad \gamma \mapsto -i\overline{\gamma} \ , \quad \overline{\gamma} \mapsto -i\gamma \ . \tag{1.8.11}$$

Clearly $\mathcal{C}^2 = 1$, $\mathcal{C}(\Phi) = \overline{\Phi}$, and $\mathcal{C}(\Gamma) = -i\overline{\Gamma}$.

Exercise 1.11. Show that the following conjugation relations hold:

$$\mathcal{C}\overline{\mathcal{D}}\mathcal{C} = -i\mathcal{D}(-1)^F \ , \qquad \mathcal{C}\mathcal{D}\mathcal{C} = -i\overline{\mathcal{D}}(-1)^F \ , \qquad \mathcal{C}\mathcal{D}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{D}\mathcal{C} \ .$$

Show that these also hold with $\mathcal{D}, \overline{\mathcal{D}} \to \mathcal{Q}, \overline{\mathcal{Q}}$.

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[4] Note that the bars now do double-duty: they distinguish the components of the Weyl fermions, and they distinguish the world-sheet complex coordinates.

Euclidean Yukawa theory

We can now easily write down the Euclidean continuation of the Yukawa theory. With our substitutions as above, we set

$$K_z = -K_{--} = \frac{1}{4}\left[\overline{\Phi}\partial\Phi - \Phi\partial\overline{\Phi}\right] - \frac{1}{2}\overline{\Gamma}\Gamma \tag{1.8.12}$$

and obtain

$$S_{\text{Euc}} = \frac{1}{2\pi}\int d^2z \left\{\mathcal{D}\overline{\mathcal{D}}\left[K_z\right] + \frac{1}{\sqrt{2}}\mathcal{D}(\Gamma J) + \frac{1}{\sqrt{2}}\overline{\mathcal{D}}(\overline{\Gamma J})\right\} . \tag{1.8.13}$$

Using $\mathcal{C}(K_z) = -K_z$ and the properties of $\mathcal{C}$ derived above it is easy to see that S_{Euc} is $\mathcal{C}$-invariant. The action has the component expansion

$$\begin{aligned} S_{\text{Euc}} = \frac{1}{2\pi}\int d^2z \Big\{&\overline{\partial\phi}\partial\phi + \overline{\psi}\partial\psi + \overline{\gamma}\bar{\partial}\gamma \\ &-\gamma J'\psi - \overline{\psi}\overline{J}'\overline{\gamma} - \overline{\gamma}E'\psi - \overline{\psi}\overline{E}'\gamma + E\overline{E} + J\overline{J}\Big\} . \end{aligned} \tag{1.8.14}$$

Exercise 1.12. Derive the superspace equations of motion that follow from S_{Euc}:

$$\mathcal{D}\Gamma - \sqrt{2}\overline{J} = 0 , \qquad \partial\mathcal{D}\Phi - \sqrt{2}\overline{J}'\overline{\Gamma} - \sqrt{2}\overline{E}'\Gamma = 0 .$$

Next, evaluate the supersymmetry variations. Define

$$\delta_\xi = \tfrac{1}{\sqrt{2}}\left[\, \xi\mathcal{Q} - \overline{\xi}\,\overline{\mathcal{Q}}\,\right] ,$$

and compute

$$\begin{aligned} \delta_\xi\phi &= \xi\psi , & \delta_\xi\psi &= -\overline{\xi}\bar{\partial}\phi , \\ \delta_\xi\overline{\phi} &= \overline{\xi}\,\overline{\psi} , & \delta_\xi\overline{\psi} &= -\xi\bar{\partial}\overline{\phi} , \\ \delta_\xi\gamma &= \xi G - \overline{\xi}E(\phi) , & \delta_\xi G &= \overline{\xi}E'(\phi)\psi - \overline{\xi}\bar{\partial}\gamma , \\ \delta_\xi\overline{\gamma} &= -\overline{\xi}\,\overline{G} + \xi\overline{E}(\phi) , & \delta_\xi\overline{G} &= \xi\overline{E}'(\phi)\overline{\psi} + \xi\bar{\partial}\overline{\gamma} . \end{aligned}$$

Note that although the supersymmetry variations depend on the holomorphic function E, the algebra closes without the use of equations of motion. Finally, using $\mathcal{C}\mathcal{Q}\mathcal{C} = -i\overline{\mathcal{Q}}(-1)^F$ show that $[\mathcal{C}, \delta_\xi] = 0$ provided we set $\mathcal{C}(\xi) = i\overline{\xi}$ and $\mathcal{C}(\overline{\xi}) = i\xi$.

References

Chapter 2

Conformalities

Abstract

%%% In this chapter we give an overview of two-dimensional conformal field theories and properties of the N=2 superconformal algebra and discuss its representations. We then introduce the moduli space and discuss some coarse invariants such as the elliptic genus, as well as some general properties of compact unitary CFTs. We begin with a quick overview of some general properties of CFTs. These are probably very familiar to most readers, but we introduce them here as a reminder and for later reference with an emphasis on the results and perspective most relevant for our (0,2) exploration. The author's favorite introduction to the subject is [8].

2.1 The basics

The conformal group in two dimensions

The global conformal group in two dimensions is $\mathrm{SO}(3,1) = \mathrm{PSL}(2,\mathbb{C})$. Its algebra is $\mathfrak{sl}_2\mathbb{C}$, and it is generated by the vector fields[1]

$$v_{-1} = -\frac{\partial}{\partial z} \ , \qquad v_0 = -z\frac{\partial}{\partial z} \ , \qquad v_1 = -z^2\frac{\partial}{\partial z} \ . \tag{2.1.1}$$

These infinitesimal transformations are realized on the operators and states of the QFT by the operators L_{-1}, L_0, , and L_1, which are, respectively, the infinitesimal generators of translations, dilatations, and special conformal transformations that satisfy the algebra

$$[L_1, L_{-1}] = 2L_0 \ , \qquad [L_0, L_{\pm 1}] = \mp L_{\pm 1} \ . \tag{2.1.2}$$

There are also the corresponding anti-holomorphic generators $\overline{L}_{0,\pm 1}$ that commute with the $L_{0,\pm 1}$. This split allows for an important simplification special to two dimensions: we can

[1]We have written the action of the generators on holomorphic functions; more generally, we have the vector fields obtained by taking $v = a_{-1}v_{-1} + a_0 v_0 + a_1 v_1 + \text{c.c.}$ for $a_{0,\pm 1} \in \mathbb{C}$.

formally regard z and $\overline{z}$ as independent variables and only impose $\overline{z} = z^*$ when we need to extract a physical observable, e.g. a correlation function.

In contrast with conformal symmetry in $d > 2$ dimensions, we also have local conformal transformations $z \to z - \epsilon(z)$ and $\overline{z} \to \overline{z} - \overline{\epsilon}(\overline{z})$ for any meromorphic functions $\epsilon(z)$ and $\overline{\epsilon}(\overline{z})$. We can therefore organize the local operators according to their transformations under these, treating z and $\overline{z}$ as independent variables. We say that a local operator $\Phi(z, \overline{z})$ is primary if under $z \to f(z)$ and $\overline{z} \to \overline{f}(\overline{z})$ it transforms as

$$\Phi(z, \overline{z}) \to \left(\tfrac{\partial f}{\partial z}\right)^{h_\Phi} \left(\tfrac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}_\Phi} \Phi(f(z), \overline{f}(\overline{z})) \ . \tag{2.1.3}$$

The pair $(h_\Phi, \overline{h}_\Phi)$ are the left- and right-moving conformal weights of the field Φ; the sum $\Delta_\Phi = h_\Phi + \overline{h}_\Phi$ is the scaling dimension (i.e. the eigenvalue of the dilatation generator), while the difference $s_\Phi = h_\Phi - \overline{h}_\Phi$ is the field's spin. An operator is quasi-primary if the transformation law above holds for $f(z), \overline{f}(\overline{z})$ restricted to global conformal transformations, i.e.

$$f(z) = \frac{az + b}{cz + d} \ , \tag{2.1.4}$$

with $ad - bc = 1$ and $(a, b, c, d) \sim (-a, -b, -c, -d)$.

Minimal assumptions

We will mostly discuss theories satisfying some "self-evident" assumptions: the class of unitary and compact CFTs.

1. The global conformal symmetry and its associated algebra $\mathfrak{sl}_2\mathbb{C}$ is represented by the current algebra constructed from components of a conserved traceless energy-momentum tensor.

2. Local fields transform in representations of $\mathfrak{sl}_2\mathbb{C}$, and each highest weight field is labeled by its spin s and scaling dimension Δ. We call these the quasi-primary fields. Any other field can be constructed as a linear combination of quasi-primary fields and their derivatives.

3. For any $\Delta_* \in \mathbb{R}$ there is a finite number of quasi-primary fields with $\Delta < \Delta_*$. A CFT with this property is called compact.

4. The theory is unitary. When defined in Minkowski space the theory will have a Hilbert space of states, a unique $\mathfrak{sl}_2\mathbb{C}$-invariant ground state $|0\rangle$, and a Hermitian Hamiltonian operator. This leads to a notion of a Hilbert space and Hermiticity in the Euclidean formulation, as we will review below.

A few comments are in order. First, note that there is a more general class of scale-invariant QFTs, where $T^\mu_\mu = \partial_\mu V^\nu$ for some virial current V; however, a unitary two-dimensional QFT with a discrete spectrum is scale invariant if and only if it is conformally invariant. [9].[2]

Second, the assumption of compactness, i.e. that there is a Hilbert space of states, excludes a number of theories. Some are "trivially non-compact," where the source of non-compactness is traced to the presence of a free scalar field; more generally, there are also interesting interacting theories, such as Liouville theory or a conformal non-linear sigma model on an ALE space. In these cases conformal invariance, and in particular conformal Ward identities for correlation functions of quasi-primary local operators, can still be used to extract quite non-trivial information about the theory; see, e.g. [11] for a discussion in the context of Liouville theory. However, some of the results that we will routinely use will not apply in this wider setting.

Finally, we should also note that non-unitary theories are also not without interest: there are also "trivial" examples of these, such as the free bc-ghost system of string theory, but there are also more interesting interacting theories, for instance various pure spinor formulations of the general type II string world-sheet [12], as well as in statistical mechanics systems without reflection positivity.

The energy-momentum tensor

We decompose the energy-momentum tensor of any Euclidean QFT with respect to the $z, \overline{z}$ coordinates as

$$\begin{aligned}
\Theta &= T_{z\overline{z}} = \tfrac{1}{4}(T_{11} + T_{22}) \ , \\
T &= T_{zz} = \tfrac{1}{4}(T_{11} - T_{22}) - \tfrac{i}{2} T_{12} \ , \\
\overline{T} &= T_{\overline{z}\overline{z}} = \tfrac{1}{4}(T_{11} - T_{22}) + \tfrac{i}{2} T_{12} \ .
\end{aligned} \tag{2.1.5}$$

The conservation equation $\partial^\mu T_{\mu\nu} = 0$ is equivalent to

$$\bar{\partial} T + \partial \Theta = 0 \ , \qquad \bar{\partial}\Theta + \partial \overline{T} = 0 \ . \tag{2.1.6}$$

A CFT has a traceless energy-momentum tensor, so that $\Theta = 0$, while T and $\overline{T}$ are separately conserved and are, respectively, holomorphic and anti-holomorphic. T and $\overline{T}$ enjoy a number of universal properties that we will now review. We will focus on T, keeping in mind that similar properties hold for $\overline{T}$.

Many CFT properties are clarified in the framework of radial quantization. Starting with our theory on $S^1 \times \mathbb{R}$ — a standard Hamiltonian formulation, we map it to the complex plane via $z = e^{i\theta+\tau}$, so that $z = 0$ corresponds to $\tau = -\infty$, while equal time slices are mapped to circles with constant $|z|$. The radial Hamiltonian is then given by

$$\boldsymbol{H} = \oint_{C(0)} \frac{dz}{2\pi i} z T(z) + \oint_{C(0)} \frac{d\overline{z}}{2\pi i} \overline{z}\,\overline{T}(\overline{z}) \ , \tag{2.1.7}$$

[2] As discussed in [10], unitarity and a discrete spectrum are both necessary for the implication to hold.

where $C(0)$ is a contour enclosing the origin and both the dz and $d\bar{z}$ integrations are oriented counter-clockwise.

Since $T(z)$ is holomorphic and of conformal weight $(2,0)$, we can expand it in modes L_n with conformal weight $(-n,0)$ according to radial quantization:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \ . \tag{2.1.8}$$

We can invert the mode expansion via

$$L_n = \oint_{C(0)} \frac{dz}{2\pi i} z^{n+1} T(z) \ , \tag{2.1.9}$$

where $C(0)$ is a contour around $z = 0$. The L_n are the generators of the infinitesimal conformal transformations, with $L_{0,\pm1}$ giving the global conformal transformations as above. Evidently the radial Hamiltonian is $\boldsymbol{H} = L_0 + \overline{L}_0$. The action of the L_n on any local operator Φ is determined by the T–Φ OPE:

$$[L_n, \Phi(w,\overline{w})] = \oint_{C(w)} \frac{dz}{2\pi i} z^{n+1} T(z)\Phi(w,\overline{w}) \ . \tag{2.1.10}$$

In particular, for a quasi-primary field, which satisfies

$$[L_1, \Phi(0,0)] = 0 \ , \qquad [L_0, \Phi(0,0)] = h\Phi(0,0) \ , \qquad [L_{-1}, \Phi(0,0)] = \partial\Phi(0,0) \ , \tag{2.1.11}$$

the OPE must take the form

$$T(z)\Phi(w,\overline{w}) \sim \cdots + \frac{A(w,\overline{w})}{(z-w)^4} + \frac{h\Phi(w,\overline{w})}{(z-w)^2} + \frac{\partial\Phi(w,\overline{w})}{(z-w)} \ , \tag{2.1.12}$$

where the $\cdots$ denote more singular terms. We say that Φ is a primary field if and only if

$$T(z)\Phi(w,\overline{w}) \sim \frac{h\Phi(w,\overline{w})}{(z-w)^2} + \frac{\partial\Phi(w,\overline{w})}{(z-w)} \ , \tag{2.1.13}$$

with the corresponding commutators

$$[L_n, \Phi(w,\overline{w})] = h(n+1)w^n\Phi(w,\overline{w}) + w^{n+1}\partial\Phi(w,\overline{w}) \ . \tag{2.1.14}$$

The global conformal algebra requires the T–T OPE to take a standard form

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \ , \tag{2.1.15}$$

where the constant c is the left (or holomorphic) central charge of the CFT, and it follows from the mode expansion that the modes L_n satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m-1)(m+1)\delta_{m,-n} \ . \tag{2.1.16}$$

Setting $c = 0$ leads to the Witt algebra — the algebra of meromorphic vector fields on the Riemann sphere. Up to a change of basis every central extension of the Witt algebra over $\mathbb{C}$ is isomorphic to the Virasoro algebra for some choice of c .%%ref.

Ward identities for T

The OPE (2.1.3), together with the conservation law $\bar{\partial}T = 0$ lead to a Ward identify for a correlation function of primary operators Φ_i:

$$\langle T(z)\Phi_1(w_1)\cdots\Phi_n(w_n)\rangle = \sum_{i=1}^{n}\left[\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i}\right]\langle\Phi_1(w_1)\cdots\Phi_n(w_n)\rangle \ . \tag{2.1.17}$$

Expanding this for $|z| \gg |w_i|$, we obtain

$$\langle T(z)\Phi_1(w_1)\cdots\Phi_n(w_n)\rangle = \left[-z^{-1}\mathcal{L}_{-1} - z^{-2}\mathcal{L}_0 - z^{-3}\mathcal{L}_1 + O(z^{-4})\right]\langle\Phi_1(w_1)\cdots\Phi_n(w_n)\rangle \ , \tag{2.1.18}$$

where

$$\mathcal{L}_{-1} = -\textstyle\sum_i \partial_{w_i} \ , \qquad \mathcal{L}_0 = -\sum_i (h_i + w_i\partial_{w_i}) \ , \qquad \mathcal{L}_1 = -\sum_i (2h_i w_i + w_i^2\partial_{w_i}) \tag{2.1.19}$$

form a representation of the $\mathfrak{sl}_2\mathbb{C}$ algebra. In view of (2.1.12), (2.1.18) holds for any quasi-primary insertions, since the more singular terms in (2.1.12) simply modify the $O(z^{-4})$ terms on the right-hand side of (2.1.18).

Exercise 2.1. Use the Ward identity above and the TT OPE to reduce the correlator $\langle T(z_1)T(z_2)\Phi_1(w_1)\cdots\Phi_n(w_n)\rangle$ to a differential operator acting on $\langle\Phi_1(w_1)\cdots\Phi_n(w_n)\rangle$. Assume the Φ_i are primary fields.

The Euclidean correlation functions have a natural interpretation in radial quantization. Once we radially order the operators $|w_1| > |w_2| > \cdots > |w_n|$, we have ,

$$\langle\Phi(w_1)\cdots\Phi(w_n)\rangle = \langle 0|\Phi(w_1)\cdots\Phi(w_n)|0\rangle \ , \tag{2.1.20}$$

where $|0\rangle$ is the unique $\mathfrak{sl}_2\mathbb{C}$–invariant vacuum state in the Hilbert space $\mathcal{H}$, while $\langle 0|$ denotes the state in the dual space $\mathcal{H}^\vee$ canonically isomorphic to $|0\rangle$. Consider (2.1.18) from this point of view. On one hand, the limit $\lim_{z\to 0} T(z)|0\rangle$ should be a state in the Hilbert space and in particular should be regular. Using our mode expansion we conclude

$$\lim_{z\to 0} T(z)|0\rangle = L_{-2}|0\rangle \tag{2.1.21}$$

if and only if

$$L_{n\geq -1}|0\rangle = 0 \ . \tag{2.1.22}$$

For $n = 0, \pm 1$ this is the familiar requirement that the vacuum should be $\mathfrak{sl}_2\mathbb{C}$-invariant, but we see that regularity implies the stronger condition.

On the other hand, in CFT we can use a special conformal transformation to map $\lim_{z\to 0} T(z)|0\rangle$ to a dual "out" state

$$\lim_{z\to\infty} z^4\langle 0|T(z) \ . \tag{2.1.23}$$

Demanding that this limit is regular leads to the requirement $\langle 0|L_n = 0$ for $n \leq 1$, and it also shows that the correlator $\langle T(z)\Phi_1(w_1)\cdots\Phi_n(w_n)\rangle$ must decay as $O(z^{-4})$ for large $|z|$. This, together with (2.1.18), implies

$$\mathcal{L}_{0,\pm 1}\langle \Phi_1(w_1)\cdots\Phi_n(w_n)\rangle = 0 \ . \tag{2.1.24}$$

These global conformal identities for correlators of quasi-primary fields determine the position dependence of two- and three-point functions. The two-point function of operators Φ and Ψ is zero unless they have equal weights, in which case

$$\langle \Phi(z,\overline{z})\Psi(0)\rangle = \frac{G}{z^{2h_1}\overline{z}^{2\overline{h}_1}} \tag{2.1.25}$$

for a constant G. The three-point function of quasi-primary fields is determined up to another constant C_{123}: [3]

$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{31}^{h_1+h_3-h_2}} \ . \tag{2.1.26}$$

Operator-state correspondence

We can generalize the construction of the state $L_{-2}|0\rangle = \lim_{z\to 0} T(z)|0\rangle$ as follows. Given a local field $\Phi(z,\overline{z})$ we define a corresponding state by

$$|\Phi\rangle = \lim_{z,\overline{z}\to 0} \Phi(z,\overline{z})|0\rangle \ , \tag{2.1.27}$$

which creates a state by a local operator insertion in the infinite past with respect to radial quantization. On the other hand, any state $|\Phi\rangle$ on the cylinder defines a corresponding local operator $\Phi(0,0)$, and since we assume the states form a Hilbert space, we conclude that there is a 1:1 map between states and operators, so that the set of local operators is also endowed with a Hilbert space structure. As we will see below, the inner product on this Hilbert space is determined by the two-point correlation functions of primary fields.

Since $L_{n\geq -1}|0\rangle = 0$ we see that the notion of a primary field is isomorphic to that of a primary state: $|\Phi\rangle$ is primary if and only if $L_{n>1}|\Phi\rangle = 0$; similarly $|\Phi\rangle$ is quasi-primary if and only if $L_1|\Phi\rangle = 0$. Equivalently, a primary state is a highest weight state of the Virasoro algebra, while a quasi-primary state is a highest weight with respect to the global conformal algebra.

Let $\mathcal{H}_{\rm qp}$ denote the Hilbert space of quasi-primary states at fixed weights:

$$\mathcal{H}_{\rm qp} = \{\lim_{z\to 0}\Phi(z,\overline{z})|0\rangle \ \mid \ \Phi \text{ is a quasi-primary with weights } (h,\overline{h})\} \ . \tag{2.1.28}$$

Denote by $\Phi^\dagger(z,\overline{z})$ the Euclidean continuation of the standard Hermitian conjugate of the Minkowski continuation of Φ. Using this, we construct a state in the dual space via

$$\langle\Phi| = \lim_{z,\overline{z}\to\infty} z^{2h}\overline{z}^{2\overline{h}}\langle 0|\Phi^\dagger(z,\overline{z}) \tag{2.1.29}$$

[3] The suppressed $\overline{z}$ dependence of the denominator can be recovered via the substitution $z \to \overline{z}$ and $h \to \overline{h}$.

The Hermitian inner product on $\mathcal{H}_{\text{qp}}$ is then determined by

$$(\Phi, \Psi) = \lim_{z,\overline{z}\to\infty} \lim_{w,\overline{w}\to 0} z^{2h}\overline{z}^{2\overline{h}} \langle 0|\Phi^\dagger(z,\overline{z})\Psi(w,\overline{w})|0\rangle = \lim_{z,\overline{z}\to\infty} z^{2h}\overline{z}^{2\overline{h}} \langle \Phi^\dagger(z,\overline{z})\Psi(0,0)\rangle \ . \quad (2.1.30)$$

Unitarity requires the two-point function on the right-hand side, known as the Zamolodchikov metric, to be positive-definite and Hermitian, so we obtain the desired inner product on $\mathcal{H}_{\text{qp}}$.

As usual, the inner product induces a notion of an adjoint operator, which we will call the conformal adjoint. This is given by

$$[\Phi(z,\overline{z})]^\circ = \overline{z}^{-2h} z^{-2\overline{h}} \Phi^\dagger(\overline{z}^{-1}, z^{-1}) \ . \quad (2.1.31)$$

The adjoint action on the coordinates is perhaps a little unfamiliar, but in fact it is a very standard notion in Euclidean QFT with a Minkowski antecedent: the definition of Euclidean time via $\tau = ix^0$ requires that we compensate the conjugation on the extra i factors with the $z \to \overline{z}^{-1}$ map. At any rate, as the following exercise shows, this conformal adjoint leads to the expected property $(\Phi, \Psi) = (\Psi, \Phi)^*$.

Exercise 2.2. Use the definition of the conformal adjoint to show that

$$(\Psi, \Phi)^* = \lim_{z,\overline{z}\to\infty} \langle \Phi^\dagger(z,\overline{z})\Psi(0,0)\rangle = (\Phi, \Psi) \ .$$

Once an operator is expanded in modes, the compatibility of the mode expansion with the conformal adjoint yields the Hermitian properties of the modes. For instance, using the mode expansion for T we have

$$[T(z)]^\circ = \left[\sum_{m\in\mathbb{Z}} L_m z^{-m-2}\right]^\circ = \sum_m L_m^\dagger (z^{-m-2})^* \ , \quad (2.1.32)$$

while from the direct definition and the fact that T is Hermitian in Minkowski space, we have

$$[T(z)]^\circ = \overline{z}^{-4} T(\overline{z}^{-1}) = \sum_m L_m \overline{z}^{m-2} \ . \quad (2.1.33)$$

Compatibility of these two forms fixes $L_m^\dagger = L_{-m}$.

Some key properties

With the basic preliminaries in hand, we can now summarize key features of unitary compact CFTs. The elementary proofs of these statements are collected in appendix A.2.

1. *The vacuum state* $|0\rangle$ *is primary and has* $(h, \overline{h}) = (0, 0)$. It corresponds to the identity operator.

2. *The central charge* c *is positive.*

3. *L_0 has a non-negative spectrum.*

4. *Every state is a sum of primary states and their Virasoro descendants.* For any primary state $|\Phi\rangle$ and an ordered partition of K $[K] = (k_1, k_2, \cdots, k_p)$, with $k_i \geq k_{i+1}$, we can define the descendant

$$L_{[K]} = L_{-k_1} L_{-k_2} \cdots L_{-k_p} |\Phi\rangle \ .$$

The statement is that every state in the Hilbert space is a linear combination of the primaries and their descendants.

5. *A local operator is anti-holormophic if and only if it has weight $h = 0$.* Of course if $h = \overline{h} = 0$ then the operator must be position-independent, and therefore (in any local quantum field theory) a constant multiple of the identity; the corresponding state is then a constant multiple of the vacuum.

Minimal models

Given any $|\Phi\rangle$ primary state with weight h, we can construct the full tower of descendants by acting with the Virasoro raising operators, which is known as a Verma module. We can organize the resulting states by their level, i.e. the integer K in the raising operators L_K defined above. For each K the result is clearly a finite-dimensional vector space, but its dimension may be smaller than the naive $P(K)$— the number of ordered partitions of K. Furthermore, the inner product on these states is completely determined by the Virasoro algebra in terms of h, the central charge c, and the norm of $|\Phi\rangle$. When $c < 1$, this inner product fails to be unitary unless

$$c = 1 - \frac{6}{m(m+1)} \ , \qquad m = 3, 4, 5, \ldots \ , \tag{2.1.34}$$

and the weight h takes one of the $\binom{m}{2}$ values determined by integers p and q:

$$h = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \ , \qquad 1 \leq q \leq p \leq m - 1 \ . \tag{2.1.35}$$

The GKO coset construction shows that a unitary CFT realizes these values of central charge and corresponding spectrum of operators [13]—these are the Virasoro minimal models.

These minimal models provide the simplest examples of solvable conformal field theories, where the principles discussed so far, together with OPE associativity and algebraic methods lead to a solution for the three-point functions of primary fields, and, more generally, for any correlation function of local operators.

A generalization of this sort of classification approach is in principle available in the class of theories known as Rational CFTs. Such theories are characterized by the presence of a chiral algebra of integer spin holomorphic currents. We will not discuss these structures and refer the reader to [14] for more details. Additional very readable discussion of RCFTs and the intimately related W-algebras is given in the excellent review [15].

Decomposable CFTs

Given two CFTs with energy momentum tensors T_1 and T_2, we can always form a direct product theory with energy-momentum tensor $T_{\rm tot} = T_1 + T_2$. The "total" theory will then have a second linearly independent conserved current with spin 2 in addition to $T_{\rm tot}$. Such constructions and their deformations of course provide many interesting examples of CFTs composed of simpler building blocks.

It is amusing to ponder a converse to the construction. In other words, suppose that we are given a CFT with a number of quasi-primary spin 2 holomorphic fields T_α, such that $T_{\rm tot} = \sum_\alpha T_\alpha$. Can we decompose the Virasoro algebra into components corresponding to the T_α? As the following example shows, there are certainly non-trivial conditions for such a decomposition to hold.

Consider the simplest case, where $\alpha = 1, 2$, and $T_{\rm tot} = T_1 + T_2$. In order to get the desired decomposition it is necessary that the OPE $T_\alpha(z_1)T_\beta(z_2)$ has the form

$$T_\alpha(z_1)T_\beta(z_2) \sim \frac{c_\alpha \delta_{\alpha\beta}}{2z_{12}^4} + \frac{2}{z_{12}^2}\delta_{\alpha\beta}T_\beta(z_2) + \frac{1}{z_{12}}\delta_{\alpha\beta}\partial T_\beta(z_2) \ . \tag{2.1.36}$$

On the other hand, if we simply demand that $T_\alpha T_\beta$ OPE closes on these operators and their descendants, $T_{\rm tot} = T_1 + T_2$, and that the modes of T_α satisfy the Jacobi identity (i.e. the four-point function has crossing symmetry) , we find that the OPE must take the form

$$T_\alpha(z_1)T_\beta(z_2) \sim \frac{G_{\alpha\beta}}{z_{12}^4} + \frac{1}{z_{12}^2}\sum_\gamma C_{\alpha\beta}{}^\gamma T_\gamma(z_2) + \tfrac{1}{2z_{12}}\sum_\gamma C_{\alpha\beta}{}^\gamma \partial T_\gamma(z_2) \ ,$$

and the constants $G_{\alpha\beta}$ and $C_{\alpha\beta}{}^\gamma$ are symmetric in α, β and satisfy

$$\sum_{\alpha,\beta} G_{\alpha\beta} = \frac{c_{\rm tot}}{2} \ , \qquad \sum_\alpha C_{\alpha\beta}{}^\gamma = 2\delta_\beta^\gamma \ , \qquad \sum_\gamma C_{\alpha\beta}{}^\gamma C_{\gamma\delta}{}^\rho = \sum_\gamma C_{\alpha\delta}{}^\gamma C_{\gamma\beta}{}^\rho \ . \tag{2.1.37}$$

There are certainly solutions to these conditions that are not compatible with decomposition. For instance, there is nothing to be done if $G_{12} \neq 0$. Even if $G_{\alpha\beta} = c_\alpha\delta_{\alpha\beta}/2$ as desired, then for any real x a solution for the $C_{\alpha\beta\gamma}$ is given by

$$C_{111} = c_1 - x \ , \qquad C_{112} = x \ , \qquad C_{122} = -x \ , \qquad C_{222} = c_2 + x \ . \tag{2.1.38}$$

Exercise 2.3. Verify the assertions in this section by studying the OPE of $T_\alpha T_\beta$ as well as the four-point function.

It should be clear that having a decomposable energy-momentum tensor leads to stronger constraints on the theory. For instance, unitarity now requires that for any primary operator the total weight $h_{\rm tot} = \sum_\alpha h_\alpha$, and the weight h_α with respect to T_α is non-negative.

2.2 Free fields

A Majorana-Weyl fermion

The simplest example of a unitary and compact CFT is provided by a single free Majorana-Weyl fermion λ with action as in (1.8.6):

$$S = \frac{1}{4\pi} \int d^2 z \, \lambda \bar{\partial} \lambda \, . \tag{2.2.1}$$

The equation of motion $\bar{\partial}\lambda = 0$ means that $\lambda(z)$ is purely holomorphic, and a standard path-integral manipulation easily produces the OPE:

$$\int_\lambda \frac{\delta}{\delta \lambda(z,\overline{z})} \left[e^{-S} \lambda(w,\overline{w}) \right] = \int_\lambda e^{-S} \left[-\tfrac{1}{2\pi} \bar{\partial}_{\overline{z}} \lambda(z,\overline{z}) \lambda(w,\overline{w}) + \delta^2(z-w) \right] \, . \tag{2.2.2}$$

The left-hand side is the integral of a total derivative; we assume that this vanishes even if we multiply $\lambda(w,\overline{w})$ by any product of local operators inserted away from $(w,\overline{w})$, and this leads to the OPE

$$\lambda(z)\lambda(w) \sim \frac{1}{z-w} \, . \tag{2.2.3}$$

From the action we see that λ has weight $(1/2, 0)$, and, indeed, using the energy-momentum tensor[4]

$$T(z) = \lim_{w \to z} \left[-\frac{1}{2} \lambda(w) \partial_z \lambda(z) - \frac{1}{2(z-w)^2} \right] = -\tfrac{1}{2} : \lambda \partial \lambda :(z) \, , \tag{2.2.4}$$

we see that

$$T(z)\lambda(w) \sim \frac{\frac{1}{2}\lambda(w)}{(z-w)^2} + \frac{\partial\lambda(w)}{z-w} \, . \tag{2.2.5}$$

We used "conformal normal ordering" to produce a well-defined local energy-momentum tensor $T(z)$. More generally, we can use the OPE structure to define products of local operators by subtracting off the singularities in the operator product.[5] In this free theory, this is just Wick's theorem:

$$\lambda(z)\lambda(w) = \frac{1}{z-w} + : \lambda(z)\lambda(w) : \, , \tag{2.2.6}$$

where the second term on the right-hand side is regular in the limit $z \to w$.

---

[4]The classical energy-momentum tensor can be obtained by the Noether procedure from the action; this can be normal-ordered to yield the operator in the quantum theory, and the normalization is fixed by the fact that T is quasi-primary of dimension 2.

[5]This is discussed systematically in [16].

Exercise 2.4. Use Wick's theorem to compute the T–T OPE and verify that the central charge is given by $c = 1/2$.

From what we said so far we have a CFT with $c = 1/2$ and $\overline{c} = 0$, and a single primary field λ of weight $(h, \overline{h}) = (1/2, 0)$. There is, however, more data that is to be determined. The fermionic field λ is of course a spinor — indeed, the spin is given by $s = h - \overline{h} = 1/2$— and therefore it need not be single-valued under $z \to e^{2\pi i} z$. On the plane, or equivalently on the cylinder, the allowed monodromies are valued in $\mathbb{Z}_2$ and are known as

$$\text{Neveu-Shwarz :} \quad \lambda(ze^{2\pi i}) = \lambda(z) \quad \text{and} \quad \text{Ramond :} \quad \lambda(ze^{2\pi i}) = -\lambda(z) \tag{2.2.7}$$

boundary conditions. We can treat these democratically by setting

$$\lambda(ze^{2\pi i}) = -e^{2\pi i\nu}\lambda \ , \tag{2.2.8}$$

where $\nu = 0$ for the R sector and $\nu = 1/2$ for the NS sector. This leads to a mode expansion

$$\lambda(z) = \sum_{r \in \mathbb{Z} - \nu} \lambda_r z^{-r-1/2} \ , \tag{2.2.9}$$

and inserting this into the OPE yields the anti-commutation relations

$$\{\lambda_r, \lambda_s\} = \delta_{r,-s} \ . \tag{2.2.10}$$

From the T–λ OPE we also learn that these modes satisfy $[L_m, \psi_r] = (m/2 - r)\psi_{r+m}$. We also have a fermion number operator $(-1)^{F_\lambda}$, which satisfies $\{\lambda(z), (-1)^{F_\lambda}\} = 0$.

Equations such as our ν-dependent boundary condition at first appear a little bit strange: what data in the quantum field theory determines which of these boundary conditions we use? The answer is that the choice of monodromy of a local field around $z = 0$ is encoded in a choice of an "in" state.[6] So, more appropriately, we consider a sub-space of the Hilbert space

$$\mathcal{H}_\nu = \{|\Psi\rangle \in \mathcal{H} \ \mid \ \lambda(ze^{2\pi i})|\Psi\rangle = -e^{2\pi i\nu}\lambda(z)|\Psi\rangle\}. \tag{2.2.11}$$

Of course without loss of generality we can decompose $\mathcal{H}_\nu$ into eigenspaces of L_0.

The canonical candidate for a state is our familiar ground state $|0\rangle$. In our free theory this is defined by demanding that $\lim_{z\to 0} \lambda(z)|0\rangle$ is regular, which means $|0\rangle \in \mathcal{H}_{\text{NS}}$. So, we have $\lambda_r|0\rangle = 0$ for $r > 0$, and $\lim_{z\to 0} \lambda(z)|0\rangle = \lambda_{-1/2}|0\rangle$. Clearly any descendant of $|0\rangle$ of the form $\lambda_{-r_1} \cdots \lambda_{-r_k}|0\rangle$ will belong to $\mathcal{H}_{\text{NS}}$, and, in fact, all states in $\mathcal{H}_{\text{NS}}$ are constructed in this fashion. For this reason it is typical to speak of $|0\rangle$ as the *NS-vacuum.* We define $(-1)^F|0\rangle = |0\rangle$, and we can decompose

$$\mathcal{H}_{\text{NS}} = \mathcal{H}_{\text{NS}}^+ \oplus \mathcal{H}_{\text{NS}}^- \tag{2.2.12}$$

[6] This is perhaps the simplest example of the more general concept of a defect in a quantum field theory; 't Hooft operators and their extensions are more sophisticated examples that occur in four-dimensional gauge theories.

based on the eigenvalue of $(-1)^{F_\lambda}$.

Evidently, the states that lead to the Ramond boundary conditions cannot be obtained from the $\mathfrak{sl}_2\mathbb{C}$–invariant vacuum by acting with any operators that are polynomial in λ and derivatives. To describe these mystery states, we first observe that they must furnish a representation of $(-1)^{F_\lambda}$ as well as the zero mode λ_0. Since $\{(-1)^{F_\lambda}, \lambda_0\} = 0$, the smallest such representation is two-dimensional, so that we must have at least a pair of states $|\Sigma_\pm\rangle$ with $L_0|\Sigma_\pm\rangle = h_\Sigma|\Sigma_\pm\rangle$. Furthermore, we will demand that $(-1)^{F_\lambda}$ is diagonalizable (otherwise it is challenging to make sense of any spin-statistics relation), so that the two-dimensional representation takes the form

$$(-1)^{F_\lambda} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} , \qquad \psi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.2.13}$$

Once we have these ground states in hand, we can obtain the remaining states in $\mathcal{H}_{\rm R}$ by acting by various raising operators $\lambda_{-r_1} \cdots \lambda_{-r_k}|\Sigma_\pm\rangle$.

The operators corresponding to the states $|\Sigma_\pm\rangle$ are known as spin fields or disorder operators, and they are characterized by a non-local OPE with the λ:[7]

$$\lambda(z)\Sigma_+(w) \sim \frac{e^{i\pi/4}}{\sqrt{2}} \frac{1}{(z-w)^{1/2}} \Sigma_-(w) . \tag{2.2.14}$$

This indicates a pathology in the CFT; we will discuss this trouble and its cure—a GSO projection—when we examine modular invariance.

Leaving the issue of mutual locality of the OPE aside, we can investigate these fields further. A useful quantity to consider are the expectation values

$$\langle 0|\lambda(z)\lambda(w)|0\rangle \qquad \langle \Sigma_+|\lambda(z)\lambda(w)|\Sigma_+\rangle . \tag{2.2.15}$$

The first one is easily evaluated from the OPE:

$$\langle 0|\lambda(z)\lambda(w)|0\rangle = \frac{1}{z-w} , \tag{2.2.16}$$

but we can also compute it by an explicit mode expansion. We have the Hermiticity condition $\lambda_r^\dagger = \lambda_{-r}$, as well as the relation $\lambda_r|0\rangle = 0$ for $r > 0$, and this leads to (for $|z| > |w|$)

$$\langle 0|\lambda(z)\lambda(w)|0\rangle = \sum_{r>0,s<0} z^{-r-1/2} w^{-s-1/2} \langle 0|\{\lambda_r, \lambda_s\}|0\rangle = \sum_{m=0}^{\infty} z^{-m-1} w^m = \frac{1}{z-w} . \tag{2.2.17}$$

Exercise 2.5. Use the mode expansion in the Ramond sector to show

$$\langle \Sigma_+|\lambda(z)\lambda(w)|\Sigma_+\rangle = \frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{z-w} .$$

Next, evaluate the weight of Σ_+ by the following trick. On one hand $T(z)|\Sigma_+\rangle = hz^{-2}|\Sigma_+\rangle$; on the other hand, we have the "point-split" definition of $T(z)$ in (2.2.4). Apply the latter to the expectation value just computed and compare with the former to obtain $h_\Sigma = 1/16$.

[7] For a derivation of the phase see [8].

The infamous scalar

The first conformal field theory that is typically encountered by a young string theorist is that of a free scalar with action

$$S = \frac{1}{4\pi} \int d^2z \, \partial\phi\bar{\partial}\phi \ . \tag{2.2.18}$$

The same sort of path integral manipulation as we carried out above for the fermion leads to the equations of motion and OPE

$$\partial\bar{\partial}\phi = 0 \ , \qquad \phi(z,\overline{z})\phi(w) \sim -\log|z-w|^2 \ . \tag{2.2.19}$$

In this case the energy-momentum tensor has non-trivial components

$$T = -\frac{1}{2}\partial\phi\partial\phi \ , \qquad \overline{T} = -\frac{1}{2}\bar{\partial}\phi\bar{\partial}\phi \ , \tag{2.2.20}$$

leading to $(c, \overline{c}) = (1, 1)$. The equation of motion allows us to separate the field into holomorphic and anti-holomorphic modes: $\phi = \phi_L(z) + \phi_R(\overline{z})$, with

$$\phi_L(z)\phi_L(w) \sim -\log(z-w) \ . \tag{2.2.21}$$

As we will see below, a single "chiral" scalar ϕ_L is a very useful formal device, even though it does not have a sensible Euclidean action.

This is a very nice free field theory, but it is not a unitary compact CFT. Clearly ϕ itself is not a quasi-primary field, and the two-point function

$$\langle\phi(z,\overline{z})\phi(0,0)\rangle = -\log|z|^2 \tag{2.2.22}$$

is neither dilatation covariant or positive. The primary operators include

$$J = i\partial\phi(z) \ , \qquad \overline{J} = i\bar{\partial}\phi(\overline{z}) \ , \qquad \mathcal{V}_k =: e^{ik\phi} : (z,\overline{z}) \ . \tag{2.2.23}$$

These have, respectively, conformal dimensions $(1, 0)$, $(0, 1)$, and $(k^2/2, k^2/2)$. In particular, we see that the $\mathfrak{sl}_2\mathbb{C}$–invariant vacuum belongs to a continuum of states labeled by the momentum $k \in \mathbb{R}$.

The diseases of this theory can be traced to the non-compact zero mode and a corresponding divergence of the Euclidean path integral. This can be cured by imposing a periodicity condition $\phi \sim \phi + 2\pi\rho$. Now ϕ is no longer a well-defined operator, and more generally, the continuum $\mathcal{V}_k$ is replaced by a discrete set of operators. We will return to this nice example a number of times.

2.3 Global symmetries

In additional to the conformal invariance a CFT may have some additional global symmetries. Suppose such a continuous symmetry is realized by a set of conserved spin 1 currents. That is, we have conserved charges

$$Q = \oint \frac{dz}{2\pi} J(z,\overline{z}) + \oint \frac{d\overline{z}}{2\pi} \overline{J}(z,\overline{z}) \tag{2.3.1}$$

that transform as scalars under the global conformal transformations and in the adjoint of a Lie algebra $\mathfrak{g}$ of the symmetry group. This requires the components J and $\overline{J}$ to have weights, respectively, $(1,0)$ and $(0,1)$ and to obey the conservation law

$$\bar{\partial} J + \partial \overline{J} = 0 \ .$$

But, as we saw above, this means that in a unitary and compact CFT we in fact have $\bar{\partial} J = 0$ and $\partial \overline{J} = 0$ separately. This has a number of implications. Most immediately, we have a doubling of the global symmetry to $\mathfrak{g}_L \oplus \mathfrak{g}_R$ for the separately conserved currents unless $J = 0$ or $\overline{J} = 0$. Let us focus for the moment on the holomorphic side corresponding to $\bar{\partial} J = 0$. The resulting structure is that of a Kac-Moody (KM), or affine Lie algebra, symmetry. Its implications for conformal field theory are discussed in the classic [17]. We will present the simplest aspects of the structure after reviewing useful properties of Lie algebras and stating our conventions.

Some Lie algebra conventions

Suppose $\mathfrak{g}$ is the Lie algebra associated to a simple compact Lie group. The structure of $\mathfrak{g}$ is fixed up to the normalization of the Cartan-Killing metric — a single constant. We will work in the normalization conventions where the longest roots have length-squared 2, and we can choose a basis of generators for the adjoint representation, which we can express in terms of explicit structure constants as

$$(t^a_{\text{adj}})^b_c = i f^{ab}{}_c \ , \tag{2.3.2}$$

where $a, b, c = 1, \ldots, \dim \mathfrak{g}$, so that

$$\text{tr}_{\text{adj}}\, t^a_{\boldsymbol{r}} t^b_{\boldsymbol{r}} = \textstyle\sum_{c,d} f^{ac}{}_d f^{bd}{}_c = 2h(\mathfrak{g})\delta^{ab} \ , \tag{2.3.3}$$

where $h(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$. This choice fixes the normalizations for the generators in every finite-dimensional unitary representation $\boldsymbol{r}$ of $\mathfrak{g}$. That is, for any representation $\boldsymbol{r}$ with Hermitian generators $t^a_{\boldsymbol{r}}$, which by definition satisfy

$$[t^a_{\boldsymbol{r}}, t^b_{\boldsymbol{r}}] = i \textstyle\sum_c f^{ab}{}_c t^c_{\boldsymbol{r}} \ , \tag{2.3.4}$$

we have

$$\text{tr}_{\boldsymbol{r}}\, t^a_{\boldsymbol{r}} t^b_{\boldsymbol{r}} = \ell(\boldsymbol{r})\delta^{ab} \ , \tag{2.3.5}$$

where $\ell(\boldsymbol{r})$ is the Dynkin index of representation $\boldsymbol{r}$. We give some values of these in the following table.

$\mathfrak{g}$	$\mathfrak{su}(n)$	$\mathfrak{so}(n)$	$\mathfrak{sp}(n)$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{f}_4$	$\mathfrak{g}_2$
$\dim\mathfrak{g}$	n^2-1	$n(n-1)/2$	$n(2n+1)$	78	133	248	52	14
$h(\mathfrak{g})=\frac{1}{2}\ell(\text{adj})$	n	$n-2$	$n+1$	12	18	30	9	4
$\ell(\text{fund})$	1	2	1	6	12	60	6	2
$\dim(\text{fund})$	n	n	$2n$	27	56	248	26	7

Here "fund" denotes the fundamental basic irreducible representation; with the exception of the spinor representations, all others basic representations are constructed from tensor products of the fundamental.[8] Note that in our conventions $\mathfrak{sp}(1)=\mathfrak{su}(2)$. For $\mathfrak{so}(n)$ we also consider the spinor representations. For $\mathfrak{so}(2k+1)=B_k$ the spinor representation W has dimension 2^k and $\ell(W)=2^{k-2}$; for $\mathfrak{so}(2k)=D_k$, each spinor representation W has dimension 2^{k-1} and $\ell(W)=2^{k-3}$. The indices of the remaining basic irreducible representations are easily computed from these by using the identities

$$\ell(\boldsymbol{r_1}\oplus\boldsymbol{r_2})=\ell(\boldsymbol{r_1})+\ell(\boldsymbol{r_2})\ , \qquad \ell(\boldsymbol{r_1}\otimes\boldsymbol{r_2})=\dim\boldsymbol{r_1}\ell(\boldsymbol{r_2})+\dim\boldsymbol{r_2}\ell(\boldsymbol{r_1})\ . \tag{2.3.6}$$

It is also useful to recall that the Dynkin index is preserved for any regular sub-algebra $\mathfrak{h}\subset\mathfrak{g}$, i.e. a sub-algebra obtained by striking nodes from the extended Dynkin diagram of $\mathfrak{g}$. In this case, for the decomposition

$$\begin{aligned}&\mathfrak{g}\supset\mathfrak{h}\\ &\boldsymbol{R}=\boldsymbol{r_1}\oplus\boldsymbol{r_2}\oplus\cdots\oplus\boldsymbol{r_n}\end{aligned} \tag{2.3.7}$$

we have $\ell_{\mathfrak{g}}(\boldsymbol{R})=\ell_{\mathfrak{h}}(\boldsymbol{r_1})+\cdots+\ell_{\mathfrak{h}}(\boldsymbol{r_n})$.[9] Finally, we note that the quadratic Casimir associated to an irreducible representation $\boldsymbol{r}$ is obtained as

$$\mathbf{C}_2(\boldsymbol{r})=\textstyle\sum_a t^a_{\boldsymbol{r}}t^a_{\boldsymbol{r}}=C_2(\boldsymbol{r})\mathbb{1}_{\boldsymbol{r}}\ , \tag{2.3.8}$$

with

$$C_2(\boldsymbol{r})=\frac{\dim\mathfrak{g}}{\dim\boldsymbol{r}}\ell(\boldsymbol{r})\ . \tag{2.3.9}$$

The Kac-Moody current algebra

We now return to our CFT with a global symmetry algebra of a simple compact Lie algebra $\mathfrak{g}$. The corresponding KM current algebra is generated by weight (1,0) currents $J^a(z)$,

[8] A final unraveling of notation: recall that a basic irreducible representation has Dynkin labels $[0,\ldots,1,\ldots,0]$.

[9] For an irregular sub-algebra the two sides are related by another integer — the index of the embedding.

$a = 1, \ldots, \dim \mathfrak{g}$. These currents have a mode expansion

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \ , \qquad J^a_n = \oint \frac{dz}{2\pi i} z^n J^a(z) \ , \tag{2.3.10}$$

so that in particular the J^a_0 are the global symmetry charges. In order for these global charges to be well-defined with respect to the $\mathfrak{sl}_2\mathbb{C}$ transformations, the $J^a(z)$ must be quasi-primary fields; unitarity then ensures that they are Virasoro primary, so that the modes satisfy

$$[L_m, J^a_n] = -nJ^a_{m+n} \ . \tag{2.3.11}$$

The global charges must also satisfy $[J^a_0, J^b_0] = i\sum_c f^{ab}{}_c J^c_0$, which fixes the OPE up to a c-number k, known as the KM level:

$$J^a(z)J^b(0) \sim \frac{k\delta^{ab}}{z^2} + \frac{if^{ab}{}_c}{z} J^c(0) \ . \tag{2.3.12}$$

The mode expansion $J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$ and the KM OPE lead to the KM algebra $\widehat{\mathfrak{g}}_k$

$$[J^a_m, J^b_n] = if^{ab}{}_c J^c_{m+n} + km\delta_{m,-n}\delta^{ab} \ . \tag{2.3.13}$$

The Hermitian adjoint for the modes is given by $(J^a_n)^\dagger = J^a_{-n}$.

The global symmetry algebra allows us to organize the states according to unitary representations of $\mathfrak{g}$, with $|\Phi_{\boldsymbol{r}}\rangle$ satisfying

$$J^a_0|\Phi_{\boldsymbol{r}}\rangle = t^a_{\boldsymbol{r}}|\Phi_{\boldsymbol{r}}\rangle \ . \tag{2.3.14}$$

A state is Kac-Moody primary if and only if it is a highest weight state with respect to the global algebra generated by the J^a_0, and it is annihilated by J^a_n for all $n > 0$ and all a. Note that the notions of Virasoro primary and KM primary states are distinct: a state can be one without being the other; we will say a state is KMV primary if it is primary with respect to both. The corresponding operators are characterized by the OPE

$$J^a(z)\Phi_{\boldsymbol{r}}(w) \sim \frac{t^a_{\boldsymbol{r}} \cdot \Phi_{\boldsymbol{r}}}{z-w} \ , \tag{2.3.15}$$

and satisfy the KM Ward identities

$$\langle J^a(z)\Phi_{\boldsymbol{r_1}}(w_1)\cdots\Phi_{\boldsymbol{r_n}}(w_n)\rangle = \sum_{i=1}^n \frac{1}{z-w_i}\langle\Phi_{\boldsymbol{r_1}}(w_1)\cdots\left[t^a_{\boldsymbol{r_i}}\cdot\Phi_{\boldsymbol{r_i}}\right]\cdots\Phi_{\boldsymbol{r_n}}(w_n)\rangle \ . \tag{2.3.16}$$

Regularity of $\lim_{z\to 0} J(z)|0\rangle$ and $\lim_{z\to\infty}\langle 0|J(z)z^2$, which is equivalent to $J^a_n|0\rangle = 0$ for $n \geq 0$, implies that the left-hand side must vanish as $O(z^{-2})$ for $|z| \gg |w_i|$, so that we obtain the expected charge conservation Ward identity

$$0 = \sum_{i=1}^n \langle\Phi_{\boldsymbol{r_1}}(w_1)\cdots\left[t^a_{\boldsymbol{r_i}}\cdot\Phi_{\boldsymbol{r_i}}\right]\cdots\Phi_{\boldsymbol{r_n}}(w_n)\rangle \ . \tag{2.3.17}$$

We now enumerate some familiar properties of KM symmetries in unitary compact CFTs (the proofs of these may be found in the references);

1. The level k is a positive integer.

2. There is a Sugawara decomposition. That is, we can write $T(z)$ as a sum of two commuting operators $T(z) = T_{KM}(z) + T'(z)$ such that $T'(z)$ has a regular OPE with the currents $J^a(z)$. The central charge of T_{KM} is

$$c_{KM} = \frac{k \dim \mathfrak{g}}{k + h(\mathfrak{g})} .$$

   It follows that the total central charge can be written as $c = c_{KM} + c'$, and unitarity requires $c' \geq 0$. Furthermore, the KM and Virasoro primary states take the form $|\Phi_{\boldsymbol{r}}\rangle \otimes |\Psi'\rangle$, where $|\Phi_{\boldsymbol{r}}\rangle$ is primary with respect to T_{KM}, and $|\Psi'\rangle$ is primary with respect to T'.

3. Every state is a linear combination of Virasoro and KM primary states, as well as their descendants.

4. Unitarity constrains the representations $\boldsymbol{r}$ and their conformal weights for any fixed level k. Restricitng attention to T_{KM}, we have

$$L_0^{\text{KM}} = \frac{1}{2(k + h(\mathfrak{g}))} \left[\sum_a J_0^a J_0^a + \sum_{n>0} \sum_a J_{-n}^a J_n^a \right] ,$$

   so that on a KM-primary state we have

$$L_0^{\text{KM}} |\Phi_{\boldsymbol{r}}\rangle = \frac{C_2(\boldsymbol{r})}{2(k + h(\mathfrak{g}))} |\Phi_{\boldsymbol{r}}\rangle = \frac{c_{\text{KM}}}{2k} \frac{\ell(\boldsymbol{r})}{\dim \boldsymbol{r}} |\Phi_{\boldsymbol{r}}\rangle .$$

   Furthermore, if λ denotes the highest weight of $\boldsymbol{r}$ and ψ the highest root of $\mathfrak{g}$, then unitarity requires $\lambda \cdot \psi \leq k$.[10] This means that in a unitary theory there is a finite number of KM primary representations at any fixed level k.

The $\widehat{\mathfrak{u}(1)}_r$ current algebra and twisting

The simplest of all KM algebras corresponds to a $\mathfrak{u}(1)$ symmetry algebra. In this case, we have just one current $J(z)$, along with KM primaries $\Phi_q(z)$ labeled by $q \in \mathbb{R}$, with the OPEs

$$J(z)J(w) \sim \frac{r}{z - w} , \qquad J(z)\Phi_q(w) \sim \frac{q\Phi_q(w)}{z - w} . \tag{2.3.18}$$

Unitarity requires $r > 0$ as long as J is not identically zero. Clearly in such a general setting there is no intrinsic way to fix the normalization of the "level" r, since by taking $J' = sJ$, we obtain $r' = s^2 r$ and $q' = sq$ for the $J'J'$ and $J'\Phi_q$ OPEs. However, suppose that $q \in \frac{1}{d}\mathbb{Z}$ for some integer d. In that case there exists a unique s such that $q' \in \mathbb{Z}$ and $\gcd(q'_1, q'_2, \ldots) = 1$.

[10] The inner product $\lambda \cdot \psi$ is computed with the Cartan-Killing metric normalized as above: $\psi \cdot \psi = 2$.

When this holds, we will fix the normalization of the current accordingly and speak of a $\widehat{\mathfrak{u}(1)}_r$ KM algebra.

The Sugawara decomposition takes a particularly simple form for a $\widehat{\mathfrak{u}(1)}_r$ KM algebra. We represent the current as $J = i\sqrt{r}\partial\phi$, where ϕ is a free chiral boson with OPE

$$\phi(z)\phi(w) \ \sim -\log(z-w) \ . \tag{2.3.19}$$

In that case, and a KMV primary operator Φ_q with weights $(h, \overline{h})$ can be decomposed as

$$\Phi_q(z) = e^{iq\phi/\sqrt{r}}(z)\widetilde{\Phi}_{h-q^2/2r,\overline{h}}(z) \ , \tag{2.3.20}$$

where $\widetilde{\Phi}$ is a $\mathfrak{u}(1)$-neutral KMV primary operator%%ref. The Sugawara energy-momentum tensor takes the form

$$T(z) = \frac{1}{2r} : JJ : (z) + T' = -\frac{1}{2} : \partial\phi\partial\phi : + T' \ , \tag{2.3.21}$$

where T' has central charge $c-1$.

Let $\mathcal{H}^0$ denote the Hilbert space of states. KMV primary states, satisfying $L_{m>0}|\Phi\rangle = 0$, $\overline{L}_{m>0}|\Phi\rangle = 0$, and $J_{m>0}|\Phi\rangle = 0$ are characterized by their weights and charges $(h, q; \overline{h})$. Given this Hilbert space, we can construct an isomorphic η-twisted Hilbert space by introducing a background charge:

$$|\eta\rangle = \lim_{z\to 0} e^{-i\eta\sqrt{r}\phi}(z)|0\rangle \ . \tag{2.3.22}$$

Evidently $J_0|\eta\rangle = -\eta r|\eta\rangle$ and $L_0|\eta\rangle = \frac{\eta^2 r}{2}|\eta\rangle$.

We call this a twist because the original fields now have a branch cut at $z = 0$:

$$\Phi_q(z)e^{-i\eta\sqrt{r}\phi}(w) \sim (z-w)^{-\eta q} : e^{iq\phi/\sqrt{r}}(z)e^{-i\eta\sqrt{r}\phi}(w) : \widetilde{\Phi}_{h-q^2/2r,\overline{h}} \ , \tag{2.3.23}$$

so that

$$\Phi_q(z^{2\pi i})|\eta\rangle = e^{-2\pi i\eta q}\Phi_q(z)|\eta\rangle \ . \tag{2.3.24}$$

It follows that if $\eta q \in \mathbb{Z}$ for all q in the spectrum, then we have returned to the original untiwsted boundary conditions. With our assumptions this holds if and only if $\eta \in \mathbb{Z}$. More generally, we now have a family of isomorphic Hilbert spaces $\mathcal{H}^\eta$ for $\eta \in [0, 1)$, where the isomorphism $\mathcal{H}^\eta = \mathcal{H}$ identifies states with $h^\eta = h - \eta q + \eta^2 r/2$ and $q^\eta = q - \eta r$. As the following exercise shows, the isomorphism between $\mathcal{H}^\eta$ and $\mathcal{H}$ respects the KM and Virasoro algebras.

Exercise 2.6. Let

$$L_n^\eta = L_n - \eta J_n + \frac{\eta^2 r}{2}\delta_{n,0} \ , \qquad J_n^\eta = J_n - \eta r\delta_{n,0} \ . \tag{2.3.25}$$

Show that the L_n^η and J_n^η realize an isomorphic algebra for every value of η.

Let us stress several features of a CFT with a $\widehat{\mathfrak{u}(1)}_r$ KM symmetry uncovered by these twisted consideratons.

1. The action of the Virasoro and KM generators on the states in a twisted sector labeled by η is represented by the L^η_n and J^η_n action on the corresponding states in the untwisted sector with $\eta = 0$.

2. Under the twist the vacuum state $|0\rangle$ is mapped to $|\eta\rangle$ with $(h, q, ; \overline{h}) = (\eta^2 r/2, -\eta r, 0)$; it follows that $\mathcal{H}$ must contain a holomorphic operator with $(h, q; \overline{h}) = (r/2, -r; 0)$, and unitarity then also requires another holormophic operator with same weight and opposite charge.

3. $\mathcal{H}$ must contain an infinite number of KMV primary fields with different q, and in particular an infinite number of holomorphic operators with $h = rk^2/2$ and $q = \pm rk$.

Note that when $\widehat{\mathfrak{u}(1)}_r$ is contained in a simple $\widehat{\mathfrak{g}}_k$ KM symmetry the latter organizes the infinite number of KMV primary representations of the former into a finite set of KMV primary representations.

Exercise 2.7. In this exercise we explore the twisting in the context of a compact scalar CFT. A compact scalar is obtained by imposing a periodicity condition $\phi \sim \phi + 2\pi\rho$. This has two effects: first, it allows us to introduce twisted sectors: $\phi(ze^{2\pi i}, \overline{z}e^{-2\pi i}) = \phi(z, \overline{z}) + 2\pi\rho$, and it quantizes the mode expansion in each sector. The primary operators are[11]

$$J = i\partial\phi \ , \qquad \overline{J} = i\bar{\partial}\phi \ , \qquad \mathcal{V}_k = e^{ik_L\phi_L + ik_R\phi_R} \ , \qquad k_{L,R} = \frac{n}{\rho} \pm \frac{m\rho^2}{2} \ ,$$

where n and m label, respectively, the momentum and winding modes, and $J, \overline{J}$ are conserved currents associated to the translation symmetry $\phi \to \phi + \text{const}$ of the scalar theory.

Show that there is a $\widehat{\mathfrak{u}(1)}_r$ KM algebra present if and only if $\rho = \sqrt{2a/b}$ for co-prime integers a, b, with $r = 2ab$. Verify that in that case the spectrum contains the holomorphic operators promised above.

2.4 The superconformal algebra

Having gone through a brief review of the basic structures, we now turn our attention to the N=2 superconformal algebra.

The global superconformal algebra

We are already familiar with its most important component — the N=2 supersymmetry algebra — from chapter 1. Modifying the notation slightly to suit our applications here, we

[11] In presenting these we are neglecting co-cycles for the $\mathcal{V}_k$; these are discussed in %%ref.

have the supercharges $G^{\pm}_{-1/2}$, as well as the translation generator L_{-1} and the R-charge J_0, which satisfy[12]

$$\{G^{\pm}_{-1/2}, G^{\pm}_{-1/2}\} = 0 \ , \qquad \{G^{+}_{-1/2}, G^{-}_{-1/2}\} = 2L_{-1} \ , \qquad [J_0, G^{\pm}_{-1/2}] = \pm G^{\pm}_{-1/2} \ . \tag{2.4.1}$$

In a conformal theory this is enlarged to a global superconformal algebra that includes additional charges L_0, L_1, and $G^{\pm}_{1/2}$. These are, respectively, the generators of dilatations, special conformal, and superconformal transformations. The non-vanishing commutators are

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n} \ , \qquad [L_m, G^{\pm}_r] = (\tfrac{m}{2} - r)G^{\pm}_{m+r} \ , \quad [J_0, G^{\pm}_r] = \pm G^{\pm}_r \ , \\ &\{G^{+}_r, G^{-}_s\} = 2L_{r+s} + (r-s)J_{r+s} \ , \end{aligned} \tag{2.4.2}$$

with $m = 0, \pm 1$ and $r, s = \pm 1/2$.

Unitary representations

The states in our CFT must form representations of this algebra, and we differentiate the states as follows. First, we see that L_0 and J_0 commute with all operators and are Hermitian with respect to the inner product defined above. We will assume that these can be diagonalized (a sensible assumption in a unitary compact theory), and we will organize operators into subspaces with fixed L_0 and J_0 eigenvalues.

A state $|\phi\rangle$ is N=2 quasi-primary if and only if

$$L_0|\phi\rangle = h_\phi|\phi\rangle \ , \qquad J_0|\phi\rangle = q_\phi|\phi\rangle \ , \qquad G^{\pm}_{1/2}|\phi\rangle = 0 \ . \tag{2.4.3}$$

Note that $|\phi\rangle$ is also a conformal quasi-primary since $L_1 = \frac{1}{2}\{G^{+}_{1/2}, G^{-}_{1/2}\}$. Among all the states we distinguish two important classes, the chiral and anti-chiral states, defined by

$$\text{chiral} \ : \quad G^{+}_{-1/2}|\phi\rangle = 0 \ , \qquad \text{anti-chiral} \ : \quad G^{-}_{-1/2}|\phi\rangle = 0 \ . \tag{2.4.4}$$

The vacuum is the unique state annihilated by all of the symmetry generators: it is quasi-primary, chiral, and anti-chiral.

Unitary superconformal representations satisfy some important constraints. The Hermitian conjugates of the generators are given by %%explain why

$$L_n^\dagger = L_{-n} \ , \qquad J_n^\dagger = J_{-n} \ , \qquad (G^{\pm}_r)^\dagger = G^{\mp}_{-r} \ . \tag{2.4.5}$$

Hence, for any state state $|\phi\rangle$ of weight and charge (h, q), we have

$$\|G^{\mp}_{1/2}|\phi\rangle\|^2 + \|G^{\pm}_{-1/2}|\phi\rangle\|^2 = (2h \mp q)\||\phi\rangle\|^2 \ . \tag{2.4.6}$$

It follows that $2h \geq \pm q$ and $h \geq 0$ for all states. The bound $2h \geq \pm q$ is saturated by two distinguished types of states:

[12] We should note a delightful clash of two reasonable uses of $\pm$ in the literature: the first denotes world-sheet chirality (we used this in the Minkowski conventions of chapter 1), and the second denotes the U(1) charges. The clash is particularly painful when working with (2,2) theories. Since we mostly work with a Euclidean world-sheet and just one copy of the N=2 algebra, we typically use the second notation.

1. chiral quasi-primary states, which satisfy $G^{\mp}_{\pm 1/2}|\phi\rangle = 0$ and $h_\phi = q_\phi/2$;
2. anti-chiral quasi-primary states, which satisfy $G^{\pm}_{\mp 1/2}|\phi\rangle = 0$ and $h_\phi = -q_\phi/2$.

Given any N=2 quasi-primary state $|\phi\rangle$, we can construct an N=2 multiplet by first acting with the raising operators $G^{\pm}_{-1/2}$ to obtain

$$|\psi^{\pm}\rangle = G^{\pm}_{-1/2}|\phi\rangle \ . \tag{2.4.7}$$

These states, if non-trivial, are conformal quasi-primary and have $(h, q) = (h_\phi + 1/2, q_\phi \pm 1)$. To complete the multiplet let

$$|X\rangle = G^{+}_{-1/2}G^{-}_{-1/2}|\phi\rangle - (1 + \tfrac{q_\phi}{2h_\phi})L_{-1}|\phi\rangle \ . \tag{2.4.8}$$

This state is constructed to satisfy $L_1|X\rangle = 0$, and the N=2 superconformal algebra closes on the "long multiplet" with conformal quasi-primary states $\{|\phi\rangle, |\psi^+\rangle, |\psi^-\rangle, |X\rangle\}$.

Exercise 2.8. Verify the previous assertion by showing

$$\begin{aligned}
&G^{+}_{-1/2}|\psi^{+}\rangle = G^{-}_{-1/2}|\psi^{-}\rangle = 0,\\
&G^{+}_{-1/2}|\psi^{-}\rangle = |X\rangle + (1 + \tfrac{q}{2h})L_{-1}|\phi\rangle, \qquad G^{-}_{-1/2}|\psi^{+}\rangle = -|X\rangle + (1 - \tfrac{q}{2h})L_{-1}|\phi\rangle,\\
&G^{+}_{1/2}|X\rangle = -\frac{(2h+q)(2h+1)}{2h}|\psi^{+}\rangle, \qquad G^{-}_{1/2}|X\rangle = \frac{(2h-q)(2h+1)}{2h}|\psi^{-}\rangle,\\
&G^{+}_{-1/2}|X\rangle = -(1 + \tfrac{q}{2h})L_{-1}|\psi^{+}\rangle, \qquad G^{-}_{-1/2}|X\rangle = (1 - \tfrac{q}{2h})L_{-1}|\psi^{-}\rangle.
\end{aligned}$$

Chiral N=2 quasi-primary states reside in shortened multiplets because the $|\psi^+\rangle$ descendant is null, so that the multiplet consists of $\{|\phi\rangle, |\psi^-\rangle\}$, with

$$\begin{aligned}
&G^{\pm}_{1/2}|\phi\rangle = 0 \ , \qquad G^{+}_{-1/2}|\phi\rangle = 0 \ , \qquad G^{-}_{-1/2}|\phi\rangle = |\psi^{-}\rangle \ ,\\
&G^{-}_{\pm 1/2}|\psi^{-}\rangle = 0 \ , \qquad G^{+}_{1/2}|\psi^{-}\rangle = 2q|\phi\rangle \ , \qquad G^{+}_{-1/2}|\psi^{-}\rangle = 2L_{-1}|\phi\rangle \ .
\end{aligned} \tag{2.4.9}$$

Of course there is an entirely analogous story for anti-chiral N=2 quasi-primary states and corresponding short multiplets $\{|\phi\rangle, |\psi^+\rangle\}$.

A little more superspace

Just as the superconformal generators act on various states in the theory, there is a corresponding action on the local operators (indeed, the two are isomorphic), leading to the notion of N=2 quasi-primary operators that reside in long or short multiplets. For instance, the operators in a chiral multiplet obey

$$\begin{aligned}
&[G^{+}_{-1/2}, \phi] = 0 \ , \qquad \{G^{+}_{-1/2}, \psi^{-}\} = 2[L_{-1}, \phi] = 2\partial\phi \ ,\\
&[G^{-}_{-1/2}, \phi] = \psi \ , \qquad [G^{-}_{-1/2}, \psi] = 0 \ .
\end{aligned} \tag{2.4.10}$$

It is useful to arrange the quasi-primary fields in superfields. To do so, we return to the Euclidean superspace discussed in the previous chapter but now apply it to our operators rather than the free fields discussed there. Since we have also been discussing a holomorphic N=2 algebra, we also send $\bar{\partial} \to \partial$ to obtain the super-derivatives and supercharges

$$\mathcal{D} = \partial_\theta + \overline{\theta}\partial \ , \qquad \mathcal{Q} = \mathcal{G}^-_{-1/2} = \partial_\theta - \overline{\theta}\partial \ ,$$
$$\overline{\mathcal{D}} = \partial_{\overline{\theta}} + \theta\partial \ , \qquad \overline{\mathcal{Q}} = \mathcal{G}^+_{-1/2} = \partial_{\overline{\theta}} - \theta\partial \ . \tag{2.4.11}$$

The components of a long multiplet can now be arranged as

$$\Phi = \phi + \theta\psi^- + \overline{\theta}\psi^+ + \theta\overline{\theta}(X + \tfrac{q}{2h}\partial\phi) \ , \tag{2.4.12}$$

and the action of the supercharges is encoded by

$$[\xi G^-_{-1/2} - \overline{\xi} G^+_{-1/2}, \Phi] = (\xi\mathcal{G}^-_{-1/2} - \overline{\xi}\mathcal{G}^+_{-1/2})\Phi \ . \tag{2.4.13}$$

In particular, the two definitions of chirality coincide. On one hand a chiral multiplet has $2h = q$ and $\psi^+ = X = 0$; on the other hand, $\overline{\mathcal{D}}\Phi = 0$ holds if and only if $X = (1 - q/2h)\partial\phi$ and $\psi^+ = 0$, in which case it takes the familiar form

$$\Phi = \phi + \theta\psi^- + \theta\overline{\theta}\partial\phi \ . \tag{2.4.14}$$

Exercise 2.9. In this exercise we develop a differential representation of the global superconformal algebra. To start, we recall the operators

$$\mathcal{L}^{(z)}_{-1} = -\partial \ , \qquad \mathcal{L}^{(z)}_0 = -z\partial \ , \qquad \mathcal{L}^{(z)}_1 = -z^2\partial$$

that represent the $\mathfrak{sl}_2\mathbb{C}$ algebra on the plane. It is easy to extend this to a representation of the global superconformal algebra on the superspace $(z, \theta, \overline{\theta})$ in a few steps. First, we already know that $\mathcal{L}_{-1} = -\partial$ and $\mathcal{G}^\pm_{-1/2}$ yield a representation of the supersymmetry algebra. Second, the R-symmetry assigns charge $+1$ to θ and -1 to $\overline{\theta}$ and leaves z invariant, so that

$$\mathcal{J}_0 = \theta\partial_\theta - \overline{\theta}\partial_{\overline{\theta}}$$

satisfies $[\mathcal{J}_0, \mathcal{G}^\pm_{-1/2}] = \pm\mathcal{G}^\pm_{-1/2}$. We also know that z and $\theta, \overline{\theta}$ have dilatation weights, respectively -1 and $-1/2$, so that

$$\mathcal{L}_0 = -z\partial - \tfrac{1}{2}(\theta\partial_\theta + \overline{\theta}\partial_{\overline{\theta}}).$$

Show that to close $\mathfrak{sl}_2\mathbb{C}$ we then need

$$\mathcal{L}_1 = -z^2\partial - z(\theta\partial_\theta + \overline{\theta}\partial_{\overline{\theta}}) \ .$$

Use this to derive the remaining generators

$$\mathcal{G}^\pm_{1/2} = [\mathcal{L}_1, \mathcal{G}^\pm_{-1/2}] = (z \mp \theta\overline{\theta})\mathcal{G}^\pm_{-1/2} \ .$$

A motivation for the superfield construction is that it gives a succinct presentation of the correlation functions for the multiplets, and the global Ward identities that follow from the superconformal invariance are realized as differential operators that annihilate the correlation functions. That is, analogously to (2.1.18) a correlation function

$$\langle \Phi_1(\boldsymbol{z}_1)\Phi_2(\boldsymbol{z}_2)\cdots\Phi_n(\boldsymbol{z}_n)\rangle \tag{2.4.15}$$

that depends on the $\boldsymbol{z}_i = (z_i, \theta_i, \overline{\theta}_i; \overline{z}_i)$ must be annihilated by the operators

$$\begin{aligned}
\mathcal{J}_0 &= \sum_i \left[\theta_i\partial_{\theta_i} - \overline{\theta}_i\partial_{\overline{\theta}_i} - q_i\right] \ , & \mathcal{L}_0 &= \sum_i \left[-h_i - z_i\partial_{z_i} - \tfrac{1}{2}(\theta_i\partial_{\theta_i} + \overline{\theta}_i\partial_{\overline{\theta}_i})\right] \ , \\
\mathcal{L}_{-1} &= \sum_i \left[-\partial_{z_i}\right] \ , & \mathcal{L}_1 &= \sum_i \left[-q\theta_i\overline{\theta}_i - 2z_ih_i - z_i^2\partial_{z_i} - z_i(\theta_i\partial_{\theta_i} + \overline{\theta}_i\partial_{\overline{\theta}_i})\right] \ , \\
\mathcal{G}^+_{-1/2} &= \sum_i \overline{\mathcal{Q}}_i \ , & \mathcal{G}^+_{1/2} &= \sum_i \left[(z_i - \theta_i\overline{\theta}_i)\overline{\mathcal{Q}}_i - (2h_i + q_i)\theta_i\right] \\
\mathcal{G}^-_{-1/2} &= \sum_i \mathcal{Q}_i \ , & \mathcal{G}^-_{1/2} &= \sum_i \left[(z_i + \theta_i\overline{\theta}_i)\mathcal{Q}_i - (2h_i - q_i)\overline{\theta}_i\right] \ .
\end{aligned} \tag{2.4.16}$$

For instance, the holomorphic dependence of the two-point function is completely determined by these identities: $\langle \Phi_1(\boldsymbol{z}_1)\Phi_2(\boldsymbol{z}_2)\rangle$ is zero unless $h_1 = h_2 = h$, $\overline{h}_1 = \overline{h}_2 = \overline{h}$, and $q_1 = -q_2 = q$, in which case it takes the form

$$\langle \Phi_1(\boldsymbol{z}_1)\Phi_2(\boldsymbol{z}_2)\rangle = \frac{C_{12}}{\zeta_{12}^{2h}\,\overline{z}_{12}^{2\overline{h}}}\left(1 - \frac{q\theta_{12}\overline{\theta}_{12}}{\zeta_{12}}\right) \ , \tag{2.4.17}$$

where

$$\zeta_{12} = z_{12} - \theta_1\overline{\theta}_2 - \overline{\theta}_1\theta_2 \ , \qquad \theta_{12} = \theta_1 - \theta_2 \ , \qquad \overline{\theta}_{12} = \overline{\theta}_1 - \overline{\theta}_2 \tag{2.4.18}$$

are supersymmetric invariants. This of course means that the two-point functions of SUSY descendants are fixed by those of the quasi-primary fields.

The super-space dependence of three-point functions of general N=2 quasi-primary operators is not fixed by superconformal invariance: there are too many independent supersymmetric invariants generalizing those that show up in the two-point function. However, the dependence is determined for three-point functions with one general quasi-primary superfield and the remaining two fields being either both chiral, both anti-chiral, or one of each chirality. In addition, three-point functions of supercurrent multiplets are also fixed; we will see an example of this shortly.

Exercise 2.10. Determine the superspace dependence of a three-point function of quasi-primary fields

$$\langle \Phi_1(\boldsymbol{z}_1)\overline{\Phi}_2(\boldsymbol{z}_2)A(\boldsymbol{z}_3)\rangle \ ,$$

where Φ and $\overline{\Phi}$ are, respectively, chiral and anti-chiral. An effective way to do this is to first find a basis for the supersymmetric invariants that can be constructed from $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3$ subject to the chirality conditions. With these in hand, one makes an Ansatz for the correlator and then uses the remaining constraints from (2.4.16). You should find that the correlation function vanishes unless $q_{\text{tot}} = q_1 + q_2 + q_A \in \{0, 1\}$. When $q_{\text{tot}} = 0$, then A must be an anti-chiral primary, in which case

$$\langle \Phi_1(\boldsymbol{z}_1)\overline{\Phi}_2(\boldsymbol{z}_2)A(\boldsymbol{z}_3)\rangle = \frac{F(\overline{z})}{\xi_{13}^{2h_1}\xi_{23}^{2h_2}} , \qquad \xi_{ik} = z_{ik} + \theta_i\overline{\theta}_{ik} + \theta_{ki}\overline{\theta}_k .$$

Check this result and then derive the expression for the case $q_{\text{tot}} = 1$.

The super-Virasoro algebra

The global superconformal algebra is a sub-algebra of the infinite-dimensional super-Virasoro algebra encoded in the conserved holomorphic currents $J(z)$, $G^{\pm}(z)$, and $T(z)$, respectively of spin 1, 3/2, and 2, with the OPEs

$$\begin{aligned} T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} , \\ T(z)G^{\pm}(w) &\sim \frac{3/2 G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{(z-w)} , \\ J(z)G^{\pm}(w) &\sim \frac{\pm G^{\pm}(w)}{z-w} . \end{aligned} \tag{2.4.19}$$

The remaining OPEs involve central terms that are fixed by closure of the algebra. The non-trivial OPEs are

$$\begin{aligned} J(z)J(w) &\sim \frac{c/3}{(z-w)^2} , \\ G^{+}(z)G^{-}(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w} , \\ T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} . \end{aligned} \tag{2.4.20}$$

Exercise 2.11. Show that these fields assemble into a supercurrent multiplet as in (1.7.5). In particular, with our superspace conventions it takes the form

$$\mathcal{S} = J + \theta G^{-} - \overline{\theta} G^{+} + 2\theta\overline{\theta} T .$$

Check that the two-point function is given by

$$\langle \mathcal{S}(\boldsymbol{z}_1)\mathcal{S}(\boldsymbol{z}_2)\rangle = \frac{c/3}{\zeta_{12}^2} .$$

Using the mode expansions

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \ , \quad J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \ , \quad G^{\pm}(z) = \sum_{r\in\mathbb{Z}\pm 1/2\pm\eta} G_r^{\pm} z^{-r-3/2} \qquad (2.4.21)$$

then leads to the mode algebra that includes the Virasoro and U(1) KM algebras, together with additional (anti)commutators

$$[L_n, G_r^{\pm}] = (\tfrac{m}{2} - r)G_{r+n}^{\pm} \ , \qquad [J_n, G_r^{\pm}] = \pm G_{r+n}^{\pm} \ , \qquad \{G_r^{\pm}, G_s^{\pm}\} = 0 \ ,$$
$$\{G_r^{+}, G_s^{-}\} = 2L_{r+s} + (r-s)J_{r+s} + \tfrac{c}{12}(4r^2-1)\delta_{r,-s} \ . \qquad (2.4.22)$$

Note that we introduced the parameter η in the moding of $G^{\pm}$, anticipating a twist by the KM algebra. The untwisted sector, where $G^{\pm}(z)|0\rangle$ is single-valued, has $\eta = 0$. As in the discussion of the twisted fermion above, we will call this the NS sector. The Hermitian adjoints of the operators take the form shown in (2.4.5), with n, r now running over the full range of values.

Sticking for now to the NS sector, we can organize the states of the CFT into muliplets of this algebra. N=2 primary states are defined to be annihilated by all the lowering modes $L_{n>0}$, $J_{n>0}$, and $G^{\pm}_{r>0}$, and all other states are obtained by acting on the primary states with raising modes. The chiral/anti-chiral quasi-primary states identified above remain distinguished in the full algebra.

Exercise 2.12. Use the N=2 super-Virasoro algebra to show that in a unitary theory a chiral (anti-chiral) state with $q = 2h$ ($-q = 2h$) is N=2 primary.

The N=2 super-Virasoro algebra has some further implications for unitarity. For instance, the norm of $G^{\mp}_{-r}$ on an N=2 primary state of weight h and charge q follows from the algebra:

$$\|G^{\mp}_{-r}|\Phi\rangle\|^2 = \big(2h \pm 2rq + \tfrac{c}{12}(4r^2-1)\big)\, \||\Phi\rangle\|^2 \ . \qquad (2.4.23)$$

Setting $r = 3/2$, we find $2h \pm 3q + 2c/3 \geq 0$, so that for chiral or anti–chiral primary states we find $h \leq c/6$.

Further implications of unitarity are discussed in [18, 19]. As in the case of the Virasoro algebra, when the central charge is sufficiently small (it turns out the value is $c < 3$) these unitarity bounds restrict the possible representations to a finite set: these are the N=2 minimal models that exist for

$$c = 3\left(1 - \frac{2}{m}\right) \ , \qquad\qquad m = 3, 4, \ldots \ . \qquad (2.4.24)$$

The chiral ring

Consider the OPE of two chiral primary operators,

$$\Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2) \sim \sum_s C_{12}{}^s z_{12}^{h_s-h_1-h_2} \overline{z}_{12}^{\overline{h}_s-\overline{h}_1-\overline{h}_2} \Psi_s(z_2, \overline{z}_2) \ . \qquad (2.4.25)$$

$\mathrm{U}(1)_{\mathrm{R}}$ charge conservation implies that only operators with $q_s = q_1 + q_2$ contribute, and this allows us to rewrite the OPE as

$$\Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2) \sim \sum_{s|q_s=q_1+q_2} C_{12}{}^{s} z_{12}^{h_s-q_s/2} \overline{z}_{12}^{\overline{h}_s-\overline{h}_1-\overline{h}_2} \Psi_s(z_2, \overline{z}_2) \ . \tag{2.4.26}$$

In a theory with (2,2) superconformal invariance we can make a further restriction to operators $\Phi \in H_{\mathrm{cc}}$, where

$$H_{\mathrm{cc}} = \{\Phi \in \mathcal{H} \ \ | \ \ h = q/2 \ \text{ and } \ \overline{h} = \overline{q}/2\} \ , \tag{2.4.27}$$

i.e. Φ is left–chiral-primary and right–chiral–primary. Since $h \leq c/6$ for a chiral primary operator, it follows that $\dim H_{\mathrm{cc}} < \infty$ in a compact CFT. Furthermore, we can grade this subspace by the charges of left- and right-moving R-symmetries $\mathfrak{u}(1)_{\mathrm{L}} \oplus \mathfrak{u}(1)_{\mathrm{R}}$:

$$H_{\mathrm{cc}} = \bigoplus_{q,\overline{q}} H_{\mathrm{cc}}^{q,\overline{q}} \ . \tag{2.4.28}$$

The OPE of operators in H_{cc} is non-singular:

$$\Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2) \sim \sum_{\substack{q_s = q_1 + q_2 \\ \overline{q}_s = \overline{q}_1 + \overline{q}_2}} C_{12}{}^{s} z_{12}^{h_s-q_s/2} \overline{z}_{12}^{\overline{h}_s-\overline{q}_s/2} \Psi_s(z_2, \overline{z}_2) \ . \tag{2.4.29}$$

In this case the limit $z_1 \to z_2$ yields another operator in H_{cc}:

$$\lim_{z_1 \to z_2} \Phi_i(z_1)\Phi_j(z_2) = C_{ij}{}^{k} \Phi_k(z_2) \ . \tag{2.4.30}$$

This endows H_{cc} with a ring structure that respects the $\mathfrak{u}(1)_{\mathrm{L}} \oplus \mathfrak{u}(1)_{\mathrm{R}}$ grading. In an analogous fashion we can also define H_{ac}, consisting of left–anti–chiral primary and right–chiral primary operators, as well as the complex-conjugate versions H_{aa} and H_{ca}.

Hodge decomposition

As we will explore in greater detail in the chapter on (0,2) geometry, there are many parallels between notions of complex geometry and the N=2 algebra. Perhaps the most basic is the statement that any state $|\phi\rangle$ can be uniquely written as a linear combination

$$|\phi\rangle = |\psi\rangle + G^+_{-1/2}|\chi\rangle + G^-_{+1/2}|\lambda\rangle \ , \tag{2.4.31}$$

where $|\psi\rangle$ is a chiral primary state. To see this, observe that our assumption of compactness guarantees that the space of chiral primary states with a fixed value of $\overline{h}$ is finite-dimensional. Working in the Hilbert space of states with a fixed $\overline{h}$, this means we have an orthogonal projector $\Pi = \Pi^\dagger$ from the Hilbert space $\mathcal{H}_{\overline{h}}$ to the subspace of chiral primary states. It follows that

$$\mathbb{1} - \Pi = (2L_0 - J_0)\Sigma \ , \tag{2.4.32}$$

where the operator Σ annihilates all chiral primary states and otherwise yields $\Sigma|\phi\rangle = (2h-q)^{-1}|\phi\rangle$, but this means

$$\mathbb{1} = \Pi + G^+_{-1/2}G^-_{1/2}\Sigma + G^-_{1/2}G^+_{-1/2}\Sigma \ . \tag{2.4.33}$$

Hence, the above expression holds with

$$|\psi\rangle = \Pi|\phi\rangle \ , \qquad |\chi\rangle = G^-_{1/2}\Sigma|\phi\rangle \ , \qquad |\lambda\rangle = G^+_{-1/2}\Sigma|\phi\rangle \ . \tag{2.4.34}$$

This decomposition into three mutually orthogonal components respects the grading by q. As we will see in chapter 4, this is entirely analogous to the Hodge decomposition for the Dolbeault operator on a compact complex manifold. %%fix connection with next paragraph!

Another parallel with geometry is in the relationship between the chiral primary states and the $G^+_{-1/2}$ cohomology. The latter can be defined as follows.

$$H_+ = \bigoplus_q H^q_+ \ , \qquad H^q_+ = \left.\frac{\{\ker G^+_{-1/2} \cap \mathcal{H}_{\overline{h}}\}}{\{\mathrm{im}\, G^+_{-1/2} \cap \mathcal{H}_{\overline{h}}\}}\right|_{\text{states with fixed } q} \ . \tag{2.4.35}$$

While $\mathcal{H}_{\overline{h}} \cap \ker G^+_{-1/2}$ is infinite-dimensional, the cohomology groups are finite-dimensional and in fact isomorphic to the space of chiral-primary operators with right-moving weight $\overline{h}$ and $\mathfrak{u}(1)_{\mathrm{R}}$ charge q. The geometric analogue is the cohomology of the Dolbeault operator $\bar\partial$ and harmonic forms of the corresponding Laplacian. In either case, the result is a straightforward consequence of the Hodge decomposition, as the reader may verify.

In the special case of (2,2) supersymmetry we can obtain decompositions like this on both the left and right. In that case we have a stronger statement: any state $|\phi\rangle$ can be decomposed as

$$|\phi\rangle = |\psi_{\mathrm{c,c}}\rangle + (G^+_{-1/2} + \overline{G}^+_{-1/2})|\chi\rangle + (G^-_{1/2} + \overline{G}^-_{1/2})|\lambda\rangle \ , \tag{2.4.36}$$

where $|\psi_{\mathrm{c,c}}\rangle$ belongs to the cc ring. This follows from exactly analogous arguments applied to the diagonal (left–right symmetric) N=2 algebra.

Spectral flow

The U(1) KM symmetry can be "bosonized" as above by writing $J = i\sqrt{\frac{c}{3}}\partial\phi$. This allows us to introduce the twisted ground states

$$|\eta\rangle = \lim_{z\to 0} e^{-i\phi\eta\sqrt{c/3}}|0\rangle \ , \tag{2.4.37}$$

and since the supercurrents $G^\pm$ are charged with respect to U(1), we see that their moding is shifted as indicated in (2.4.21). In particular, for $\eta = 0$, (the NS sector), $G^\pm(z)$ are single-valued. More generally, we have an isomorphism of the different twisted Hilbert spaces $\mathcal{H}^\nu$, and this isomorphism respects the N=2 algebra: a simple generalization of exercise 2.6 shows that we can represent the action of the N=2 algebra in a twisted sector labeled by η by an isomorphic algebra defined on the untwisted sector with generators

$$L_n^\eta = L_n - \eta J_n + \frac{\eta^2 c}{6}\delta_{n,0} \ , \qquad J_n^\eta = J_n - \frac{\eta c}{3}\delta_{n,0} \ , \qquad G_r^{\eta\pm} = G^\pm_{r\mp\eta} \ , \qquad r \in \mathbb{Z} \pm \eta \pm 1/2 \ . \tag{2.4.38}$$

This equivalence goes by the name of "spectral flow." [20, 21].

As we saw in our general discussion of twisting, quantization of the U(1) charges has important implications. So it is here. Consider, first, the situation where the charges with respect to J of the N=2 algebra are quantized. In this case we have a natural isomorphism from the NS sector to itself obtained by "1 unit of spectral flow," i.e. $\eta = 1$. Under this identification the chiral primary states with charge q and weight $h = q/2$ are mapped to

$$(q, q/2) \xrightarrow{\eta=1} (q - c/3, -(q - c/3)/2) \ . \tag{2.4.39}$$

Thus, we observe that

1. in such theories $c/3 \in \mathbb{Z}$;
2. the space of chiral primary states is isomorphic to that of anti-chiral primary states, with an "order-reversing" isomorphism $q \to c/3 - q$;
3. there exists a holomorphic anti-chiral primary operator $\overline{\Omega}(z)$ with $q = -c/3$ that is the image of the vacuum. Its conjugate is the chiral primary $\Omega(z)$ with $q = c/3$).

The last point implies that the theory has an extended chiral algebra; this structure is explored further in [22].

Note that the operators are $\mathbb{Z}_2$ graded with respect to the ± 1 eigenvalues of $e^{i\pi J_0}$, a grading reminiscent of that by a fermion number operator $(-1)^F$. This is more than a coincidence: in most theories that we will discuss we will be able to identify $e^{i\pi J_0}$ as a "left-moving" contribution to the fermion number. In these sorts of theories, the flow by 1/2 unit of spectral flow leads to the Ramond sector, which has some important features. A key point is that now the supercurrents have zero-modes $G_0^\pm$, which satisfy

$$\{G_0^+, G_0^-\} = 2\left(L_0 - \frac{c}{24}\right) = 2E_0 \ . \tag{2.4.40}$$

We define the Ramond ground states as the states annihilated by $G_0^\pm$.[13] Obviously these states have a vanishing ground state energy E_0, and we can identify them via (2.4.38) with

[13] The off-set of $-c/24$ in the ground state energy E_0 has a nice interpretation: if we make a conformal transformation to go from the plane to the cylinder via $z = e^w$, we obtain a standard Hamiltonian formulation of our theory on $S^1 \times \mathbb{R}$, with $\mathbb{R}$ specifying the Euclidean time. In making this transformation, $T(z)$ picks up an inhomogeneous shift by a Schwartzian derivative which leads to a shift of the modes: $(L_0)_{\text{cyl}} = L_0 - c/24$.

the corresponding states in the NS sector: since $G_0^{\eta\pm} = G^{\pm}_{\mp 1/2}$, the Ramond ground states are in 1:1 correspondence with the chiral primary states: a chiral primary state with charge q corresponds to a Ramond ground state with charge $q - c/6$.

Rational charges

More generally, we might have a situation where the J charges are rational, so that for some choice of level $r \in \mathbb{Z}$, not necessarily with $r = c/3$, we have a $\widehat{\mathfrak{u}(1)}_r$ KM algebra of the sort defined above. In this case, we will still have the natural identification of the untwisted sector with $\mathcal{H}^1$, as well as the corresponding holomorphic operators with charge $\pm r$.

2.5 N=1,3,4

The N=2 algebra, whether superconformal or merely supersymmetric, obviously plays a key role in (0,2) QFT. However, other two-dimensional supersymmetries are also occasionally useful, and in this section we will make a brief tour of these structures.

Every N=2 algebra has an N=1 sub-algebra with a supercurrent

$$G(z) = \frac{1}{\sqrt{2}} G^+(z) + \frac{1}{\sqrt{2}} G^-(z) \ , \tag{2.5.1}$$

and therefore OPE

$$G(z)G(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w} \ . \tag{2.5.2}$$

Of course we can be a bit more general and take $G(z) = \frac{1}{\sqrt{2}} [e^{i\alpha} G^+(z) + e^{-i\alpha} G^-(z)]$ for any phase α. The N=1 algebra is of course much less constraining than the N=2 algebra, but it is quite interesting because of the key role it plays in superstrings and heterotic strings, where it appears as a fundamental symmetry of the RNS string. In particular, the most general background of the critical perturbative heterotic string corresponds to a choice of an N=1 superconformal theory! That being said, it is perhaps not surprising that it is not an easy class of theories to tackle in full generality.

The remaining possibilities are classified once we make some reasonable assumptions:

1. the energy momentum tensor is the unique field with spin 2;

2. the remaining fields have spins $s \in \{1/2, 1, 3/2\}$;

3. the algebra contains the N-extended supersymmetry algebra

   $$\{G^A_{-1/2}, G^B_{-1/2}\} = 2\delta^{AB} L_{-1}$$

   as a subalgebra;

4. the Virasoro algebra is a sub-algebra as well.

Remarkably, there is just one possibility without "extra" spin 1/2 fields: the (small) N=4 algebra. This contains a $\widehat{\mathfrak{su}(2)}_k$ KM algebra and has central charge $c = 6k$. The 4 supercharges transform as $\mathbf{2} \oplus \mathbf{2}$ of the $\mathfrak{su}(2)$.

In addition, there are two possibilities with spin 1/2 holomorphic fields ψ: the N=3 and the large N=4 algebra. The former has an $\widehat{\mathfrak{su}(2)}_k$ KM and $c = 3k/2$, with supercharges in $\mathbf{3}$ of $\mathfrak{su}(2)$ and one $\mathfrak{su}(2)$–invariant free fermion. The large N=4 algebra has the KM symmetry

$$\widehat{\mathfrak{su}(2)}_{k_1} \oplus \widehat{\mathfrak{su}(2)}_{k_2} \oplus \mathfrak{u}(1) \ ,$$

and central charge $c = 6k_1k_2/(k_1 + k_2)$. The supercharges and the fermions ψ transform in $(\mathbf{2}, \mathbf{2})_0$ representation. We will not consider these possibilities in more detail here, but more information may be found in [20, 23, 24]. On the other hand, we will return to the (small) N=4 algebra in due course and will drop the diminutive terminology from here on.

Exercise 2.13. In this exercise we take a look at extended supersymmetry realizations via free fields. Consider a Lagrangian for a free real compact scalar ϕ and a Majorana-Weyl fermion λ

$$\mathcal{L} = \partial\phi\bar{\partial}\phi + \lambda\bar{\partial}\lambda \ .$$

Verify that this is invariant under the SUSY transformation

$$Q \cdot \phi = i\lambda \qquad\qquad Q \cdot \lambda = -i\partial\phi \ .$$

Now consider n copies of the same theory, with

$$\mathcal{L} = \partial\phi^T\bar{\partial}\phi + \lambda^T\bar{\partial}\lambda \ .$$

Suppose there is an additional invariance

$$Q' \cdot \phi = i\mathcal{J}\lambda \ , \qquad\qquad Q' \cdot \lambda = i\mathcal{K}\partial\phi \ , \tag{2.5.3}$$

such that $(Q')^2 = \partial$ and $\{Q', Q\} = 0$. Show that $\mathcal{K} = \mathcal{J}$, and $\mathcal{J}$ is an anti-symmetric matrix satisfying $\mathcal{J}^2 = -\mathbb{1}_n$. Such a $\mathcal{J}$ defines a Hermitian structure on the Euclidean target-space T^n where ϕ takes values; evidently, such a $\mathcal{J}$ exists if and only if $n = 2m$. Thus, we obtain N=2 (more precisely (2,0)) supersymmetry on a complex manifold.

Next, consider a more general situation, where we have a number $N - 1$ of such extra supersymmetries, Q_A, corresponding to $N - 1$ complex structures $\mathcal{J}_A$. In order to obtain a standard SUSY algebra

$$\{Q_A, Q_B\} = 2\delta_{AB}\partial \ ,$$

we need the complex structures to satisfy $\{\mathcal{J}_A, \mathcal{J}_B\} = -2\delta_{AB}\mathbb{1}_n$. Show that given two complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$, we automatically obtain a third linearly independent one, $\mathcal{J}_3 = \mathcal{J}_1\mathcal{J}_2$. Thus, in this case, we see that as soon as we have $N = 3$, we actually have

$N = 4$, and it corresponds to a hyper-Hermitian structure, characterized by $\mathcal{J}_A$, $A = 1, 2, 3$ that obey

$$\mathcal{J}_A \mathcal{J}_B = -\delta_{AB} + \textstyle\sum_C \epsilon_{ABC} \mathcal{J}_C \ .$$

Show that such a structure is possible if and only if $n = 4k$.

Finally, show that $N \geq 5$ is too nice to exist: there is no $\mathcal{J}_4$. This can be accomplished by evaluating

$$(\mathcal{J}_1 \mathcal{J}_2 \mathcal{J}_3 \mathcal{J}_4)^2$$

in two different ways, once using the anti-commutation relations of all the $\mathcal{J}$, and once using $\mathcal{J}_3 = \mathcal{J}_1 \mathcal{J}_2$.

The example has one more little lesson. While, of course, there is at most one N=4 supersymmetry algebra that closes to translations, the action is invariant under many more fermionic transformations simply because it is a sum of free supersymmetric theories. We will return to this when we tackle the geometry of non-linear sigma models.

Superconformal Kac-Moody algebras

Suppose that a theory with an N=1 superconformal algebra has a KM current algebra with currents J^A. In a unitary theory we can easily show that the J^A can be decomposed into two sets of commuting currents. We summarize the results here, most of which can be found in Chapter 18 of [10], and leave the details for the appendix.

The currents in the first set, which we will denote by $\{J^\alpha\}_{\alpha=1,\ldots,\dim\mathfrak{g}}$, generate a KM algebra $\widehat{\mathfrak{g}}_k$ and are top components of N=1 multiplets; the corresponding lowest components, Ψ^α, are N=1 primary fermionic operators with $h = 1/2$. These N=1 multiplets $\{(\Psi^\alpha, J^\alpha)\}$ form an N=1 superconformal current algebra: the degrees of freedom can be recast into a set of bosonic KM currents J_b^α with level $k_b = k - h(\mathfrak{g})$ and a set of free fermions transforming in the adjoint representation of $\mathfrak{g}$. Note that the latter lead to an additional KM algebra, $\mathfrak{so}\widehat{(2\dim\mathfrak{g})}_1$. All together, the Ψ^α and J^α degrees of freedom contribute

$$c_{\text{SKM}} = \frac{k_b \dim\mathfrak{g}}{k_b + h(\mathfrak{g})} + \frac{1}{2}\dim\mathfrak{g} = \left(\frac{k_b}{k} + \frac{1}{2}\right)\dim\mathfrak{g} \ . \tag{2.5.4}$$

The second set of currents, $\{J^a\}$ will not commute with the supercharge; in fact, they will have an OPE

$$J^a(z)G(w) = \frac{\mathcal{X}^a(w)}{z-w} \ , \tag{2.5.5}$$

where the $\mathcal{X}^a$ will be holomorphic operators with $h = 3/2$. The Jacobi identity can be used to show that these extend the superconformal algebra, and with the assumptions above we see that the J^a must then generate the corresponding R-symmetry algebra.

The structure of KM algebras in N=2 SCFTs is just a little bit more restricted. As discussed in [24, 25], the key requirement is that the compact simply connected Lie group of $\mathfrak{g}$ must admit an integrable complex structure compatible with the metric. We will return to this point again when we study (0,2) non-linear sigma models, but for now we will just state the key fact: any even dimensional compact Lie group admits an integrable complex structure [26, 27]. This means that an N=1 SKM algebra corresponding to an even-dimensional compact Lie group is automatically an N=2 SKM. Finally, we note that just as the KM algebra has a Sugawara construction, so do the N=1 and N=2 SKMs. [25, 28, 29] That means that any time we have a theory with an $N = 2$ SKM, the superconformal algebra decomposes into two commuting factors:

$$\mathcal{A}_c = \mathcal{A}_{c-c_{\text{SKM}}} \oplus \mathcal{A}_{c_{\text{SKM}}} \ . \tag{2.5.6}$$

Exercise 2.14. The simplest N=2 SKM algebra has $c = 3$ and consists of a free Weyl fermion ψ, its conjugate $\overline{\psi}$, and two bosonic currents combined into complex combinations K and $K^\dagger$. The non-vanishing OPEs of these fields take the form

$$\psi(z)\psi^\dagger(w) \sim \frac{1}{z-w} \ , \qquad\qquad K(z)K^\dagger(w) \sim \frac{1}{(z-w)^2} \ .$$

Show that the following fields satisfy the $c = 3$ $N = 2$ algebra:

$$\begin{aligned} &J_{\text{skm}} =: \psi\psi^\dagger : \ , && G_{\text{skm}} = \sqrt{2}\psi K^\dagger \ , \quad \overline{G}_{\text{skm}} = \sqrt{2}\psi^\dagger K \ , \\ &T_{\text{skm}} =: KK^\dagger : \ - \tfrac{1}{2}(: \psi^\dagger\partial\psi : + : \psi\partial\psi^\dagger :) \ . \end{aligned}$$

Next, show that $J' = J - J_{\text{skm}}$ and similarly defined G' $\overline{G}'$ and T' commute with the N=2 SKM fields.

2.6 Background fields

The properties of a QFT are often elucidated by coupling it to background fields. This is particularly powerful when the coupling is made by gauging a continuous symmetry of the theory. In this section we will explore such gaugings for a CFT.

Gauging a KM symmetry

Suppose we have a CFT with a $\widehat{\mathfrak{u}(1)}_k \oplus \widehat{\mathfrak{u}(1)}_{\overline{k}}$ symmetry and corresponding currents $J(z)$ and $\overline{J}(\overline{z})$ satisfying

$$\langle J(z)J(0)\rangle = \frac{k}{z^2} \ , \qquad\qquad \langle \overline{J}(\overline{z})\overline{J}(0)\rangle = \frac{\overline{k}}{\overline{z}^2} \ . \tag{2.6.1}$$

We introduce a background gauge field, with components A and $\overline{A}$, and we define the QFT in the background via

$$\langle \cdots \rangle_{A,\overline{A}} = \langle \cdots e^{-S[A,\overline{A}]}\rangle \ , \tag{2.6.2}$$

where $\cdots$ represents the insertions of any local operators and

$$S[A,\overline{A}] = \frac{1}{2\pi}\int d^2z\,\left\{J\overline{A} + \overline{J}A\right\}\ . \tag{2.6.3}$$

The correlation functions to any order in the background are then obtained by expanding the exponential. Of course the right-hand-side is not well-defined as written. Since the action involves an integral over the world-sheet there will be singularities when the interaction terms approach the insertions or each other; moreover, since the OPE is only well-defined for separated points, there will also be ambiguities and corresponding choices of local counter-terms. These issues are addressed by fixing some regularization scheme and renormalizing the operators and S.

In our example the story is rather simple, since the correlators of the currents satisfy a Wick's theorem, e.g., if we denote $J(z_i)$ by J_i, we have

$$\begin{gathered}\langle J_1J_2\rangle = \frac{k}{z_{12}^2}\ , \qquad \langle J_1J_2J_3\rangle = 0\ , \\ \langle J_1J_2J_3J_4\rangle = \langle J_1J_2\rangle\langle J_3J_4\rangle + \langle J_1J_3\rangle\langle J_2J_4\rangle + \langle J_1J_4\rangle\langle J_2J_3\rangle\ .\end{gathered} \tag{2.6.4}$$

So, if we define the partition function

$$Z[A,\overline{A}] = \langle 1\rangle_{A,\overline{A}} = \langle e^{-S[A,\overline{A}]}\rangle\ , \tag{2.6.5}$$

we have $Z = e^{-W}$ with

$$W[A,\overline{A}] = \frac{1}{8\pi^2}\int d^2z_1d^2z_2\left\{k\frac{\overline{A}_1\overline{A}_2}{z_{12}^2} + \overline{k}\frac{A_1A_2}{\overline{z}_{12}^2}\right\}\ . \tag{2.6.6}$$

The integrand exhibits a mild singularity as $z_1 \to z_2$, which we can dispose of by defining the improper integral as $\lim_{\ell\to 0}\int_{|z_{12}|\geq\ell}$. While simple, W is not gauge-invariant. The original action $S[A,\overline{A}]$ is invariant under

$$\delta_f A = -\partial f\ , \qquad\qquad \delta_f\overline{A} = -\bar{\partial}f \tag{2.6.7}$$

for any function $f(z,\overline{z})$ because J and $\overline{J}$ are (separately) conserved currents. On the other hand, using (1.8.2) we have

$$\begin{aligned}\delta W &= -\frac{1}{2\pi}\int d^2z\ f\left\{k\partial\overline{A} + \overline{k}\bar{\partial}A\right\} \\ &= -\frac{1}{4\pi}(k+\overline{k})\delta_f\int d^2zA\overline{A} - \frac{1}{4\pi}(k-\overline{k})\int d^2z\ f(\partial\overline{A} - \bar{\partial}A) \\ &= -\frac{k+\overline{k}}{4\pi}\delta_f\int d^2zA\overline{A} - \frac{i(k-\overline{k})}{4\pi}\int fF\ ,\end{aligned} \tag{2.6.8}$$

where $F = F_{12}dy^1\wedge dy^2 = -2i(\partial\overline{A} - \bar{\partial}A)dy^1\wedge dy^2$ is the gauge-invariant field strength.

Exercise 2.15. The quick way to make this computation is simply by ignoring the ℓ regulator and using (1.8.2). Show that the boundary terms in the regulated integral lead to the same form for the gauge variation. This of course had to be: the gauge variation is guaranteed to be local and finite.

While the first contribution is manifestly the gauge variation of a local operator, the second contribution cannot be put in this form. Hence, gauge invariance can only be maintained if $k = \overline{k}$. If that holds, we add the counter-term to obtain the gauge-invariant

$$W[A, \overline{A}] = \frac{k}{8\pi^2} \int d^2 z_1 d^2 z_2 \left\{ \frac{\overline{A}_1 \overline{A}_2}{z_{12}^2} + \frac{A_1 A_2}{\overline{z}_{12}^2} \right\} + \frac{k}{2\pi} \int d^2 z A \overline{A} \ . \tag{2.6.9}$$

Indeed, we evaluate

$$\begin{aligned} \langle J_1 \rangle_A = 2\pi \frac{\delta W}{\delta \overline{A}_1} &= -\frac{k}{2\pi} \int d^2 z_2 \frac{\partial_2 \overline{A}_2}{z_{12}} + k A_1 \ , \\ \langle \overline{J}_1 \rangle_A = 2\pi \frac{\delta W}{\delta A_1} &= -\frac{k}{2\pi} \int d^2 z_2 \frac{\bar{\partial}_2 A_2}{\overline{z}_{12}} + k \overline{A}_1 \ . \end{aligned} \tag{2.6.10}$$

So, the current has a non-zero 1-point function in the background, but it remains conserved:

$$\bar{\partial}_1 \langle J_1 \rangle_A + \partial_1 \langle \overline{J}_1 \rangle_A = 0 \ . \tag{2.6.11}$$

Although the partition function does not have the same simple form, the same conclusion holds true for non-abelian KM algebras as well: we can preserve gauge invariance by adding a local counter-term if and only if $k = \overline{k}$.

There is another, equivalent, perspective on the origin of this contact term: the OPEs we used in the computation are only determined for separated points, and there are ambiguities in the contact terms that arise when operators collide. These are constrained by physical requirements, such as gauge invariance. The analysis above amounts to the observation that if $k = \overline{k}$, then we can preserve gauge invariance by introducing a contact term

$$\langle J(z_1) \overline{J}(\overline{z}_2) \rangle = k \delta^2(z_{12}, \overline{z}_{12}) \ . \tag{2.6.12}$$

Note that this is consistent with the scale invariance of the CFT; more generally, the absence of a scale in the original theory constrains the possible contact terms. When we regulate a non-chiral theory, for instance when we gauge the rotation symmetry of a free complex scalar, such a contact term will be "automatically" provided in the regulated theory.

Exercise 2.16. Check gauge invariance in the non-abelian case by expanding the partition function up to $O(A^4, \overline{A}^4)$ corrections for a general KM algebra with currents satisfying (2.3.12).

A background metric

While we have so far discussed the theory defined on the Euclidean plane, we can use the energy-momentum tensor to couple the theory to a background metric $g = \delta + h$, where h is a small perturbation, by following the same strategy as we used for the gauge field. For any theory, not necessarily a CFT, we write

$$S[h] = -\frac{1}{8\pi}\int d^2z\ h^{\mu\nu}T_{\mu\nu}\ , \tag{2.6.13}$$

where h is the metric perturbation, and indices are raised/lowered with the background flat metric. Writing this in terms of the Θ, T and $\overline{T}$ components as in (2.1.5), we obtain

$$\begin{aligned} S[h] &= -\frac{1}{8\pi}\int d^2z\left[\tfrac{1}{2}(h_{11}+h_{22})(T_{11}+T_{22}) + \tfrac{1}{2}(h_{11}-h_{22})(T_{11}-T_{22}) + 2h_{12}T_{12}\right] \\ &= -\frac{1}{8\pi}\int d^2z\left[\ \sigma\Theta + \bar{\mathsf{h}}T + \mathsf{h}\overline{T}\ \right]\ , \end{aligned} \tag{2.6.14}$$

where, repeating (2.1.5) for convenience

$$\Theta = \tfrac{1}{4}(T_{11}+T_{22})\ , \qquad T = \tfrac{1}{4}(T_{11}-T_{22}) - \tfrac{i}{2}T_{12}\ , \qquad \overline{T} = \tfrac{1}{4}(T_{11}-T_{22}) + \tfrac{i}{2}T_{12}\ ,$$

and

$$\sigma = 2(h_{11}+h_{22})\ , \qquad \bar{\mathsf{h}} = h_{11}-h_{22}+2ih_{12}\ , \qquad \mathsf{h} = h_{11}-h_{22}-2ih_{12}\ . \tag{2.6.15}$$

Returning now to the CFT, where, of course, $\Theta = 0$ we define the perturbed correlators via

$$\langle\cdots\rangle_h = \langle\cdots e^{-S[h]}\rangle\ . \tag{2.6.16}$$

Using the T–T OPE we therefore obtain to leading order in the perturbation

$$\langle T_1\rangle_h = \frac{c}{96\pi}\int d^2z_2\frac{\partial_2^3\bar{\mathsf{h}}_2}{z_{12}}\ , \qquad\qquad \langle\overline{T}_1\rangle_h = \frac{\bar{c}}{96\pi}\int d^2z_2\frac{\bar{\partial}_2^3\mathsf{h}_2}{\bar{z}_{12}}\ . \tag{2.6.17}$$

This does not bode well for conservation of the energy-momentum tensor in a general background, but we have yet to consider the possible local counter-terms. These must be Lorentz-invariant and contain two derivatives; we restrict attention to terms quadratic in the perturbation since we work to first order in the 1-point functions. This leads to the following possible terms:

$$S_{\text{c.t.}} = -\frac{1}{8\pi}\int d^2z\left[a_1\bar{\mathsf{h}}\partial^2\sigma + a_2\bar{\mathsf{h}}\partial\bar{\partial}\mathsf{h} + a_3\mathsf{h}\bar{\partial}^2\sigma + a_4\sigma\partial\bar{\partial}\sigma\right] \tag{2.6.18}$$

for some to-be-determined coefficients a_1, a_2, a_3, a_4.

Exercise 2.17. Show that the inclusion of $S_{\text{c.t.}}$ leads to

$$\begin{aligned}
\langle T\rangle_h &= \frac{c}{96\pi}\int d^2z_2 \frac{\partial_2^3\overline{\mathsf{h}}_2}{z-z_2} + a_1\partial^2\sigma + a_2\partial\bar\partial\mathsf{h} \ ,\\
\langle\Theta\rangle_h &= a_1\partial^2\overline{\mathsf{h}} + a_3\bar\partial^2\mathsf{h} + 2a_4\partial\bar\partial\sigma \ ,\\
\langle\overline{T}\rangle_h &= \frac{\bar c}{96\pi}\int d^2z_2 \frac{\bar\partial_2^3\mathsf{h}_2}{\bar z-\bar z_2} + a_3\bar\partial^2\sigma + a_2\partial\bar\partial\,\overline{\mathsf{h}} \ .
\end{aligned}$$

Show that these satisfy the conservation equations (2.1.6) if and only if $c = \bar c$, in which case

$$a_1 = -a_2 = a_3 = -2a_4 = -c/48 \ ,$$

which in turn leads to

$$\langle\Theta\rangle_h = \frac{c}{48}\left[\partial\bar\partial\sigma - \partial^2\overline{\mathsf{h}} - \bar\partial^2\mathsf{h}\right] \ .$$

From the exercise it follows that $c = \bar c$ is necessary in order to preserve diffeomorphism invariance, and, unless $c = \bar c = 0$, conformal invariance is then violated; we recognize the violation as proportional to the linearized Ricci-scalar, so that the fully covariant term is uniquely determined (the derivative order is fixed by scale invariance) to be[14]

$$\langle\Theta\rangle_g = -\frac{c}{48}R(g) \ . \tag{2.6.19}$$

This violation is the two-dimensional version of the famous conformal anomaly, and we see that it is the price to pay for preserving diffeomorphism invariance. Quite generally, in any even dimension there are two types of contributions denoted as type A and B. The former is essentially topological in character and arises from a part of the effective action W that does not involve any scale μ; the latter is associated to terms in W that involve a renormalization scale explicitly. There are no type B anomalies in two dimensions. A lucid account and classification of A and B anomalies is provided in [30].

Incidentally, there is another kind of central term that occasionally shows up in discussing world-sheet anomalies. It may be that a (1,0) current J has OPE with T

$$T(z)J(w) \sim \frac{a}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} \ , \tag{2.6.20}$$

and if we were to couple T and J to background fields, then the coefficient a would lead to a mixed anomaly. Such an anomaly can always be removed by improving the energy-momentum tensor. Namely, if $J(z)J(w) \sim r(z-w)^{-2}$, then setting $T_{\text{new}} = T + \frac{a}{2r}\partial J$ leads to a standard Virasoro-KM algebra for J and T_{new} with central charge $c_{\text{new}} = c + 3a^2/2r$.

[14] We recall that for $g = \delta + h$ the linearized form of the Ricci scalar is $R(g) = -\partial^\mu\partial_\mu h^\alpha_\alpha - \partial^\alpha\partial^\beta h_{\alpha\beta}$.

2.7 Conformal perturbation theory

We now turn to one of the most important concepts in quantum field theory in general: the renormalization group flow and QFTs defined by conformal perturbation theory. On the one hand, this is a central feature in our conceptual understanding of QFT; on the other hand, it is technically quite challenging to provide a complete and computable definition of this framework. The discussion will focus on two dimensional Lorentz-invariant theories, but the essential points apply in any dimension and to less symmetric situations.

The idea, in a nut-shell, is that given a CFT for which we know the spectrum of local operators $\mathcal{O}_i(z,\overline{z})$ with spins $s_i = h_i - \overline{h}_i$ and scaling dimensions $\Delta_i = h_i + \overline{h}_i$, and their correlation functions, we can attempt to define a perturbed theory by

$$\langle \cdots \rangle_\lambda = \langle \cdots e^{-S[\lambda]} \rangle \ , \qquad\qquad S[\lambda] = \int d^2 z \sum_i \lambda^i \mathcal{O}_i(z,\overline{z}) \ . \tag{2.7.1}$$

To preserve Lorentz invariance we restrict the sum to operators with $s_i = 0$; evidently λ^i has a mass dimension $\eta_i = 2 - \Delta_i$. If we know all of the correlation functions of the unperturbed theory, then we can obtain all correlation functions in the perturbed theory by expanding the exponential order by order in λ and computing the integrals.

The nut-shell neglects a crucial subtlety: the right-hand side is not defined due to singularities as the integrated operators collide with each other and with the insertions (the operators in $\cdots$). A concrete regulator is provided by cutting out "little disks" of radius a from the domain of integration. In general we must also address potential IR divergences. This may be done in a number of ways, for instance by placing the theory on a compact round sphere S^2 of radius L. Alternatively, we may take the couplings to be background fields $g^i(z,\overline{z})$ with compact support, or we can introduce an explicit position space cut-off L and restrict the integrated operators to $|z| \leq L$.

While the regulated correlation functions are well-defined (though quite ponderous to compute even at relatively low orders in perturbation theory), they also depend on the UV regulator. To obtain a local QFT from this description, we must renormalize the theory. In the next section we will take a look at what the result of renormalization should be.

Renormalized QFT

In this section we will remind the reader what sort of structure we would like from a renormalized conformal perturbation theory. Our discussion follows closely what is given in [31]. An alternative (and more powerful) approach makes stronger assumptions based on a local renormalization group formulation [32–36]. While our main interest is in two-dimensional theories, much of what we say here is dimension–independent.

We assume that a renormalized QFT includes the following ingredients.

1. We consider (the Euclidean continuation of) a unitary, Lorentz-invariant QFT.

2. There is a set of local operators $\widehat{\Phi}_A(w,\overline{w})$ of mass dimension Δ_A. The m-point correlation functions $\langle \widehat{\Phi}_{A_1}(w_1,\overline{w}_1)\cdots\widehat{\Phi}_{A_m}(w_m,\overline{w}_m)\rangle$ are the basic observables. Whenever

it is not confusing, we will use the shorthand $\widehat{\Phi}_{A_i} = \widehat{\Phi}_{A_i}(w_i, \overline{w}_i)$. For simplicity we will assume that the $\widehat{\Phi}_A$ transform as primary operators with respect to scale transformations: under $(w, \overline{w}) \to (sw, s\overline{w})$ we have $\widehat{\Phi}_A(w, \overline{w}) \to s^{-\Delta_A}\widehat{\Phi}_A(sw, s\overline{w})$.

3. The correlation functions of the local operators depend on a renormalization length-scale ℓ and a (possibly infinite) set of dimensionless couplings g^i, but the dependence is subject to the Callan-Symanzik equation:

$$\left\{-\ell\frac{\partial}{\partial\ell} + \sum_i \beta^i(g)\frac{\partial}{\partial g^i}\right\}\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle = \Gamma\cdot\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle \ , \tag{2.7.2}$$

where the β function is

$$\beta^i(g) = -\ell\frac{dg^i}{d\ell} \ , \tag{2.7.3}$$

and

$$\begin{aligned}\Gamma\cdot\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle = \Gamma^B_{A_1}\langle\widehat{\Phi}_B(z_1)\widehat{\Phi}_{A_2}\cdots\widehat{\Phi}_{A_m}\rangle + \Gamma^B_{A_2}\langle\widehat{\Phi}_{A_1}\widehat{\Phi}_B(z_2)\widehat{\Phi}_{A_3}\cdots\widehat{\Phi}_{A_m}\rangle \\ + \cdots + \Gamma^B_{A_m}\langle\widehat{\Phi}_{A_1}\widehat{\Phi}_{A_m}\cdots\widehat{\Phi}_B(z_m)\rangle \ .\end{aligned} \tag{2.7.4}$$

$\Gamma^A_B(g)$ is the matrix of "anomalous dimensions."[15] The Callan-Symanzik equation is the key assumption: it guarantees that the dependence on the renormalization scale can be absorbed into a change of the couplings determined by the β function and a change of basis for the local operators. Note that β and Γ have no explicit ℓ dependence, and by dimensional analysis $\Gamma^B_A = 0$ whenever $\Delta_A \neq \Delta_B$.

4. The local operators include a conserved energy-momentum tensor, and its trace Θ generates scale transformations. That is, if we set

$$\mathcal{S} = \sum_{s=1}^m\left[w_s\frac{\partial}{\partial w_s} + \overline{w}_s\frac{\partial}{\partial\overline{w}_s}\right] \ , \qquad \Delta_{\text{tot}} = \sum_{s=1}^m \Delta_{A_s} \ , \tag{2.7.5}$$

then %%why the stupid π?

$$\left\{\mathcal{S} + \Delta_{\text{tot}} - \widehat{\Delta}\right\}\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle = -\frac{1}{\pi}\int_{\text{reg}} d^2z\langle\Theta(z,\overline{z})\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle \ . \tag{2.7.6}$$

The $\widehat{\Delta}$ is a linear operator that acts in the same way as Γ. Its motivation comes from the divergences in the integral on the right-hand-side: the integral needs to be regulated, and $\widehat{\Delta}$ provides the counterterms necessary to remove the cut-off dependence. As we will see, even when $\Theta = 0$, so that the renormalized theory is in fact a conformal theory, there is indeed a non-trivial $\widehat{\Delta}$ contribution.

[15]In this section we will frequently use the summation convention: repeated indices are summed. We hope this is clear from context.

All of the correlation functions obey a tautological identity. Since we are free to change our reference lengthscale (i.e. our meter stick), we have

$$\left\{\mathcal{S} + \Delta_{\text{tot}} + \ell\frac{\partial}{\partial\ell}\right\}\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle = 0 \ . \tag{2.7.7}$$

This encodes the trivial fact that we are free to change our reference length-scale. It does lead to an interesting equation when combined with (2.7.6):

$$-\ell\frac{\partial}{\partial\ell}\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle = \widehat{\Delta}\cdot\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle - \frac{1}{\pi}\int_{\text{reg}} d^2z\langle\Theta(z,\bar{z})\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle \ . \tag{2.7.8}$$

We can also combine this with the Callan-Symanzik equation:

$$\beta^i\frac{\partial}{\partial g^i}\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle - \frac{1}{\pi}\int_{\text{reg}} d^2z\langle\Theta(z,\bar{z})\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle = (\Gamma-\widehat{\Delta})\cdot\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle \ . \tag{2.7.9}$$

A family of conformal field theories

Suppose we deform a CFT_0 by exactly marginal operators to produce a family of theories CFT_g that depend on the dimensionless couplings. By assumption we then have $\beta = 0$ and $\Theta = 0$, and (2.7.9) requires $\widehat{\Delta} = \Gamma$.

As we deform the theory by changing g, the exactly marginal operators will retain their $g = 0$ mass dimension, namely $\Delta = 0$. However, the dimensions of other operators will depend on g. The compact boson of exercise 2.7 is a good concrete example to keep in mind: the marginal coupling is a function of the radius ρ, the exactly marginal operator is just $\partial\phi\bar{\partial}\phi$, and the dimensions of the exponentials vary smoothly with ρ. We will now see that when we consider the CFT family as a renormalized QFT, these shifts in the dimensions are captured by the Γ.[16]

To get a sense for how this works it is sufficient to focus on two-point functions of operators in the renormalized QFT, which by dimensional analysis have the form

$$\langle\widehat{\Phi}_A(x)\widehat{\Phi}_B(0)\rangle = x^{-\Delta_A-\Delta_B}G_{AB}(x/\ell;g) \ , \tag{2.7.10}$$

where G_{AB} is a symmetric (and in a unitary theory positive-definite) matrix. Since $\Gamma_A^C(g) = 0$ unless $\Delta_A = \Delta_C$, the Callan-Symanzik equation implies

$$-\ell\frac{\partial}{\partial\ell}G_{AB} = \Gamma_A^C G_{CB} + G_{AC}\Gamma_B^C \ . \tag{2.7.11}$$

The equation has the solution

$$G_{AB}(\ell/x;g) = \left(e^{-\Gamma\log\ell/x}\widetilde{G}(g)e^{-\Gamma^T\log\ell/x}\right)_{AB} \ , \tag{2.7.12}$$

[16] This is analogous to what occurs in perturbation theory around a free theory, where operators acquire anomalous dimensions that differ from the engineering dimensions of the classical fields and depend on the couplings.

where the symmetric matrix $\widetilde{G}_{AB}$ is

$$\widetilde{G}_{AB}(g) = G_{AB}(1;g) = \ell^{\Delta_A+\Delta_B}\langle\widehat{\Phi}_A(\ell)\widehat{\Phi}_B(0)\rangle \ . \tag{2.7.13}$$

Suppose that Γ is diagonalizable, i.e.

$$\Gamma = S^{-1}DS \ , \tag{2.7.14}$$

where $S(g)$ is an invertible linear transformation, and

$$D = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_N) \ . \tag{2.7.15}$$

Let $\mathcal{G} = SGS^T$. From the general solution for G we then find

$$\mathcal{G}_{AB}(\ell/x;g) = \left(\frac{x}{\ell}\right)^{\gamma_A+\gamma_B}\mathcal{G}_{AB}(1;g) \ , \tag{2.7.16}$$

Therefore, the ℓ dependence of the two-point functions can be completely absorbed into a rescaling of the fields. More precisely, if we set $\Psi_A = \ell^{\gamma_A}S_A^B\widehat{\Phi}_B$, we obtain

$$\langle\Psi_A(x)\Psi_B(0)\rangle = x^{-\Delta_A-\Delta_B+\gamma_A+\gamma_B}\mathcal{G}_{AB}(1;g) \ . \tag{2.7.17}$$

This is exactly what we expect to find in a family of conformal theories: the operator dimensions shift away from the $g = 0$ values Δ_A to $\Delta_A - \gamma_A(g)$, and renormalized operators have power-law correlation functions consistent with the operator dimensions. We expect that the dilatation operator should be Hermitian in a unitary compact CFT, and therefore the assumption that Γ is diagonalizable is reasonable.

There is more to conformal invariance than scale invariance, and were we to write down the Ward identities for T and $\overline{T}$, which are separately conserved because $\Theta = 0$, we would be able to recover the further constraints on two-point and three-point functions.

The action principle and geometry

We will make another assumption on the structure of a renormalized QFT. This assumption is sometimes called the action principle, and it is motivated by the path integral source formulation of Lagrangian field theories. We postulate that, whether the theory has a Lagrangian description or not, the coupling dependence of the correlation functions is constrained by

$$\frac{\partial}{\partial g^i}\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle = B_i(g,\ell/a)\cdot\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle - \int_{\mathrm{reg}} d^2z\langle\widehat{\mathcal{O}}_i(z,\overline{z})\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle \ . \tag{2.7.18}$$

That is, to every coupling we associate a local operator $\widehat{\mathcal{O}}_i$, and there is a linear operator B_i such that (2.7.18) holds for all correlation functions. We call the $\widehat{\mathcal{O}}_i$ the deforming operators, and we assume that the set $\{\widehat{\mathcal{O}}_i\}_{i\in I}$ is complete in the sense that

$$(B_i)^A_j = 0 \ , \qquad \Gamma^A_i = 0 \ , \qquad \widehat{\Delta}^A_i = 0 \tag{2.7.19}$$

unless $A \in I$. We will continue to use this informal index notation: $A, B, C, \ldots$ label all the operators; while $i, j, k, \ldots$ label the deforming operators.

A key point is that, unlike Γ, the B_i and $\widehat{\Delta}$ can have an explicit ℓ-dependence—this follows from the divergences in the integrals. We emphasize this by writing an explicit cut-off a in (2.7.18); the equation is to be understood in the $a \to 0$ limit, but this still leaves a possible ℓ-dependence.

We note that for simplicity we are omitting an important possibility in the mixing of the operators. In general we must allow for the $\widehat{\mathcal{O}}_i$ to mix with total derivatives, e.g. there may be relations of the form $c^i \widehat{\mathcal{O}}_i = \partial_\mu J^\mu$. Such linear combinations of operators do not have couplings associated to them, but they can and do mix with the deforming operators. They go by the general term of "redundant operators," and a detailed discussion relevant to two-dimensions is given in [37].

Finally, we postulate a geometric structure for the coupling dependence of our theory. That is, the g^i can be thought of as coordinates on some manifold $\mathcal{M}$. In general $\mathcal{M}$ may be infinite-dimensional, but in favorable cases, when the theory is renormalizable, it is finite dimensional. We assume that the correlation functions transform under diffeomorphisms on $\mathcal{M}$ as sections of various bundles over $\mathcal{M}$. For instance, m-point correlators of the deforming operators have a natural interpretation as sections of $(T^*_{\mathcal{M}})^{\otimes m}$, and if we think of $\langle \widehat{\Phi}_{A_1} \cdots \widehat{\Phi}_{A_m} \rangle$ as a section of a bundle $\mathcal{E} \to \mathcal{M}$, then we have an interpretation for the action principle, which we rewrite as

$$\mathcal{D}_i \langle \widehat{\Phi}_{A_1} \cdots \widehat{\Phi}_{A_m} \rangle = \left(\frac{\partial}{\partial g^i} - B_i \right) \langle \widehat{\Phi}_{A_1} \cdots \widehat{\Phi}_{A_m} \rangle = - \int_{\text{reg}} d^2 z \langle \widehat{\mathcal{O}}_i \widehat{\Phi}_{A_1} \cdots \widehat{\Phi}_{A_m} \rangle \ . \tag{2.7.20}$$

That is, we now have a derivative operator $\mathcal{D}$ that is a map $\mathcal{E} \to \mathcal{E} \otimes T^*_{\mathcal{M}}$. At this point there is certainly no "natural" choice of connection B_i—we expect this to depend on the precise renormalization conditions we choose for our theory.

The action principle constrains the relation between Θ and the deforming operators. To see this, we use (2.7.20) in (2.7.9) to find

$$- \int_{\text{reg}} d^2 z \langle \left[\pi \beta^i \widehat{\mathcal{O}}_i(z, \overline{z}) + \Theta(z, \overline{z}) \right] \widehat{\Phi}_{A_1} \cdots \widehat{\Phi}_{A_m} \rangle = \pi (\Gamma - \widehat{\Delta} - \beta^i B_i) \cdot \langle \widehat{\Phi}_{A_1} \cdots \widehat{\Phi}_{A_m} \rangle \ . \tag{2.7.21}$$

Since this holds in arbitrary correlation functions, it is tempting to conclude that there must be an operator relation

$$\pi \beta^i \widehat{\mathcal{O}}_i + \Theta = \partial_z(\cdots) + \partial_{\overline{z}}(\cdots) \ , \tag{2.7.22}$$

which holds unless the left-hand-side is placed at a coincident point with another insertion; in that case the right-hand-side of (2.7.21) may be understood as arising from a contact term. In explicit examples (2.7.22) is always satisfied, with the total derivatives arising as a consequence of the redundant operators and relations of the form $c^i \widehat{\mathcal{O}}_i = \partial_\mu J^\mu$ mentioned above. It is also possible to argue for this equation in the framework of the local renormalization group. However, it is not obvious to the author how to argue for (2.7.22) from (2.7.21) and the general assumptions made so far.

Properties of the B connection

We will now discuss some aspects of the connection B on the coupling space. Our first point is cautionary: the divergent integrals that occur in the relevant definitions must be treated carefully. For instance, one might naively conclude that the connection is flat, i.e. the covariant derivative $\mathcal{D}_i$ satisfies $[\mathcal{D}_i, \mathcal{D}_j] = 0$. After all, formally

$$\begin{aligned}\mathcal{D}_i\mathcal{D}_j\langle\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle &= -\mathcal{D}_i\int d^2z\langle\widehat{\mathcal{O}}_j(z,\overline{z})\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle \\ &= \int d^2yd^2z\langle\widehat{\mathcal{O}}_i(y,\overline{y})\widehat{\mathcal{O}}_j(z,\overline{z})\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}\rangle \ , \end{aligned} \tag{2.7.23}$$

and the last line appears to be symmetric in i, j. This is not the case precisely because the integral is formal, and in general the connection has curvature. For instance, the moduli space of a CFT is certainly not flat in general [34, 38].[17]

We will now discuss some properties of the B connection; the reader may wish to review some of the geometric concepts in appendix B.2. To simplify the presentation, we will abbreviate the rather ponderous $\widehat{\Phi}_{A_1}\cdots\widehat{\Phi}_{A_m}$ by X_A; the results extend over multiple indices in an obvious multilinear form. We will also often abbreviate $\frac{\partial}{\partial g^i}$ by ∂_i.

First, we define the torsion tensor

$$T^k_{ij} = B^k_{ij} - B^k_{ji} \ . \tag{2.7.24}$$

By our assumption of the completeness of the deforming operators, we have no occasion to speak of T^A_{ij} with $A \notin I$. The commutator of the derivatives yields the curvature tensor F:

$$\begin{aligned}[\mathcal{D}_m, \mathcal{D}_i]\langle X_A\rangle &= \left(\partial_i B^C_{mA} - \partial_m B^C_{iA} + B^B_{mA}B^C_{iB} - B^B_{iA}B^C_{mB}\right)\langle X_C\rangle + (B^k_{im} - B^k_{mi})\mathcal{D}_k\langle X_A\rangle \\ &= (F_{im})^C_A\langle X_C\rangle + T^k_{im}\mathcal{D}_k\langle X_A\rangle \ , \end{aligned} \tag{2.7.25}$$

where the curvature tensor is defined as

$$(F_{im})^C_A = \partial_i B^C_{mA} - \partial_m B^C_{iA} + B^B_{mA}B^C_{iB} - B^B_{iA}B^C_{mB} \ . \tag{2.7.26}$$

The curvature satisfies a Bianchi identity:

$$(F_{im})^k_j + (F_{ji})^k_m + (F_{mj})^k_i = \mathcal{D}_i T^k_{mj} + \mathcal{D}_j T^k_{im} + \mathcal{D}_m T^k_{ji} \ . \tag{2.7.27}$$

Exercise 2.18. Use the covariant derivative to rewrite the Callan-Symanzik equation in a covariant form:

$$-\ell\frac{\partial}{\partial\ell}\langle X_A\rangle = -\beta^m\mathcal{D}_m\langle X_A\rangle + S^B_A\langle X_B\rangle \ , \tag{2.7.28}$$

[17] In the context of (2,2) SCFT topological field theory techniques may be used to evaluate the moduli space metric [39], and at least locally the metric is known in many non-trivial examples. A pedagogical general discussion of moduli space geometry from a CFT point of view and applications to moduli spaces of (2,2) SCFTs may be found in [40]; in the same context it has been shown that sphere partition functions may be used to compute the moduli space metric and also derive general constraints on the global moduli space geometry [41].

where S is given by

$$S_A^B = \Gamma_A^B - \beta^m B_{mA}^B \ . \tag{2.7.29}$$

Argue that for a sensible geometric interpretation S should transform as a tensor under coupling-space diffeomorphisms (this is not the case for either Γ or B separately.).

We require that the partial derivatives with respect to ℓ and g^i commute, i.e.

$$[-\ell\frac{\partial}{\partial\ell}, \frac{\partial}{\partial g^i}]\langle X_A\rangle = 0 \ . \tag{2.7.30}$$

This leads to a consistency requirement that is explored in the next exercise.

Exercise 2.19. Let

$$\mathcal{X}_j^i = \Gamma_j^i + \partial_j\beta^i \ . \tag{2.7.31}$$

Show that $\mathcal{X}_j^i$ is a tensor by rewriting it as

$$\mathcal{X}_j^i = S_j^i + \mathcal{D}_j\beta^i + \beta^m T_{mj}^i \ . \tag{2.7.32}$$

Next, show that setting $[-\ell\frac{\partial}{\partial\ell}, \frac{\partial}{\partial g^i}]\langle X_A\rangle = 0$ leads to

$$\mathcal{X}_i^j\mathcal{D}_j\langle X_A\rangle = \left\{\beta^m(F_{im})_A^B + \mathcal{D}_i(S_A^B) + \left[\ell\frac{\partial}{\partial\ell}B_{iA}^B\right]\right\}\langle X_B\rangle \ . \tag{2.7.33}$$

We recast the result of the exercise as

$$-\int_{\text{reg}} d^2z\langle(\Gamma_i^j + \partial_i\beta^j)\widehat{\mathcal{O}}_j(z,\bar{z})X_A\rangle = \left\{\beta^m(F_{im})_A^B + \mathcal{D}_i(S_A^B) + \left[\ell\frac{\partial}{\partial\ell}B_{iA}^B\right]\right\}\langle X_B\rangle \ . \tag{2.7.34}$$

Just as in our discussion of (2.7.21) and (2.7.22), it is tempting to conclude that this requires an operator relation

$$(\partial_j\beta^i + \Gamma_j^i)\widehat{\mathcal{O}}_i = \partial_z(\cdots) + \partial_{\bar{z}}(\cdots) \ , \tag{2.7.35}$$

and this again holds in explicit examples. Indeed, as we will see in the next section, modulo subtleties of redundant operators, one finds the stronger conditions

$$\begin{aligned} 0 &= \partial_j\beta^i + \Gamma_j^i \ , \\ 0 &= \beta^m(F_{im})_A^B + \mathcal{D}_i(S_A^B) + \left[\ell\frac{\partial}{\partial\ell}B_{iA}^B\right] \ . \end{aligned} \tag{2.7.36}$$

Bare and renormalized couplings

Conformal perturbation theory offers an example, indeed a model, of the general structure we discussed above. We will now illustrate the general structure in this "more hands-on" setting. This will still be formal, as we will not specify specific regularization and renormalization prescriptions, but it will give insight into the features described above. To simplify notation we will only discuss correlation functions of the deforming operators and assume there are no redundant operators.

A theory CFT_0 has local spinless operators $\mathcal{O}_i$, and we suppose we know all of their correlation functions. We then define renormalized correlation functions for a deformed theory as follows:

$$\langle \widehat{\mathcal{O}}_{k_1} \cdots \widehat{\mathcal{O}}_{k_n} \rangle = \lim_{a \to 0} \langle \widehat{\mathcal{O}}_{k_1} \cdots \widehat{\mathcal{O}}_{k_n} e^{-\int d^2z \lambda_B^i \mathcal{O}_i} \rangle_{\mathrm{reg}} \ . \tag{2.7.37}$$

Here a is a UV cut-off, and the subscript "reg" on the right-hand-side denotes some regularization prescription (such as cutting out little disks). The operator $\mathcal{O}_i$ has scaling dimension Δ_i, and we set $\eta_i = 2 - \Delta_k$. By dimensional analysis the bare couplings take the form

$$\lambda_B^i = \ell^{-\eta_i} \Lambda_B^i(g, \ell/a) \ , \tag{2.7.38}$$

where ℓ is a renormalization scale, and g^i are the dimensionless renormalized couplings. To leading order in conformal perturbation theory $\Lambda_B^i = g^i$, so that at least in vicinity of $g = 0$ these functions are invertible.

Finally, the renormalized operators are related by linear transformations to the bare ones:

$$\widehat{\mathcal{O}}_i = Z_i^j(g, \ell/a) \ell^{-\eta_j} \mathcal{O}_j \ , \qquad \Longleftrightarrow \qquad \mathcal{O}_j = \ell^{\eta_j} (Z^{-1})_j^k \widehat{\mathcal{O}}_k \ . \tag{2.7.39}$$

The Z_i^j will be invertible in perturbation theory; with this definition the $\widehat{\mathcal{O}}_i$ have mass dimension 2.

The β function is determined by demanding that the bare couplings are independent of the renormalization scale:

$$0 = \ell \frac{d\lambda_B^i}{d\ell} = \ell \frac{\partial \lambda_B^i}{\partial \ell} + \frac{\partial \lambda_B^i}{\partial g^k} \ell \frac{dg^k}{d\ell} = \ell \frac{\partial \lambda_B^i}{\partial \ell} - \frac{\partial \lambda_B^i}{\partial g^k} \beta^k \ . \tag{2.7.40}$$

We solve this for the β function:

$$\beta^k = \left(\frac{\partial g}{\partial \lambda_B} \right)_i^k \ell \frac{\partial \lambda_B^i}{\partial \ell} \ . \tag{2.7.41}$$

The Callan-Symanzik equation

We compute

$$\begin{aligned} \left(-\ell \frac{\partial}{\partial \ell} + \beta^j \frac{\partial}{\partial g^j} \right) \langle \widehat{\mathcal{O}}_{i_1} \cdots \widehat{\mathcal{O}}_{i_n} \rangle &= -\ell \frac{d}{d\ell} \left[Z_{i_1}^{j_1} \cdots Z_{i_n}^{j_n} \ell^{-\eta_{j_1} - \cdots - \eta_{j_n}} \right] \langle \mathcal{O}_{j_1} \cdots \mathcal{O}_{j_n} \rangle \\ &= \Gamma \cdot \langle \widehat{\mathcal{O}}_{i_1} \cdots \widehat{\mathcal{O}}_{i_n} \rangle \ , \end{aligned} \tag{2.7.42}$$

where

$$\Gamma_i^j \widehat{\mathcal{O}}_j = -\ell \frac{d}{d\ell}\left(Z_i^k \ell^{-\eta_k}\right)\mathcal{O}_k \qquad \Longleftrightarrow \qquad \Gamma_j^i = -\ell \frac{d}{d\ell}\left(\ell^{-\eta_k} Z_j^k\right)\ell^{\eta_k}(Z^{-1})_k^i \ . \tag{2.7.43}$$

The action principle

We compute

$$\frac{\partial}{\partial g^j}\langle \widehat{\mathcal{O}}_{i_1}\cdots\widehat{\mathcal{O}}_{i_n}\rangle = B_j \cdot \langle \widehat{\mathcal{O}}_{i_1}\cdots\widehat{\mathcal{O}}_{i_n}\rangle - \int d^2z \frac{\partial \lambda_B^k}{\partial g^j}(\ell^{\eta_k} Z^{-1})_k^m \langle \widehat{\mathcal{O}}_m(z)\widehat{\mathcal{O}}_{i_1}\cdots\widehat{\mathcal{O}}_{i_n}\rangle \ , \tag{2.7.44}$$

where

$$(B_j)_i^k = \frac{\partial Z_i^m}{\partial g^j}(Z^{-1})_m^k \ . \tag{2.7.45}$$

So, for the action principle to hold as stated above, we need

$$Z_j^k = \ell^{\eta_k}\frac{\partial \lambda_B^k}{\partial g^j} \ . \tag{2.7.46}$$

When this holds we see that B is a torsion-free connection, and $\mathcal{X}_i^m = 0$. To prove the latter statement we use (2.7.46) in (2.7.43):

$$\Gamma_j^i = -\ell \frac{d}{d\ell}\left(\frac{\lambda_B^k}{\partial g^j}\right)\ell^{\eta_k}(Z^{-1})_k^i \ . \tag{2.7.47}$$

Now we commute derivatives:

$$-\ell \frac{d}{d\ell}\frac{\partial}{\partial g^j} = \left(-\ell \frac{\partial}{\partial \ell} + \beta^m \frac{\partial}{\partial g^m}\right)\frac{\partial}{\partial g^j} = -\partial_j \beta^m \frac{\partial}{\partial g^m} + \frac{\partial}{\partial g^m}\left(-\ell \frac{d}{d\ell}\right) \ . \tag{2.7.48}$$

Since $d\lambda_B/d\ell = 0$, it now follows

$$\Gamma_j^i = -\partial_j \beta^m \frac{\partial \lambda_B^k}{\partial g^m}\ell^{\eta_k}(Z^{-1})_k^i = -\partial_j \beta^m \ . \tag{2.7.49}$$

The B connection

Since $\mathcal{X} = 0$, to be consistent with general renormalized QFT we need the connection B to satisfy (2.7.36).

Exercise 2.20. Use

$$B_{ji}^k = \frac{\partial Z_i^m}{\partial g^j}(Z^{-1})_m^k = \frac{\partial^2 \lambda_B^m}{\partial g^i \partial g^j}\left(\frac{\partial g}{\partial \lambda_B}\right)_m^k \tag{2.7.50}$$

to show that, as promised,

$$-\ell \frac{\partial}{\partial \ell} B_{ij}^k = \beta^m (F_{im})_j^k - \mathcal{D}_i \mathcal{D}_j \beta^k \ . \tag{2.7.51}$$

Now we study the consequences of the relation for B for the conditions we arrived at generally above. To simplify the computation we will liberally use $B^k_{ij} = B^k_{ji}$. We have

$$B^k_{ji} = \frac{\partial Z^m_i}{\partial g^j}(Z^{-1})^k_m = \frac{\partial^2 \lambda^m_B}{\partial g^i \partial g^j}\left(\frac{\partial g}{\partial \lambda_B}\right)^k_m . \tag{2.7.52}$$

From this we derive

$$-\ell\frac{\partial}{\partial \ell}B^k_{ji} = \frac{\partial^2}{\partial g^i \partial g^j}\left(-\ell\frac{\partial \lambda^m_B}{\partial \ell}\right)\left(\frac{\partial g}{\partial \lambda_B}\right)^k_m - B^n_{ij}\frac{\partial}{\partial g^n}\left[-\ell\frac{\partial \lambda^m_B}{\partial \ell}\right]\left(\frac{\partial g}{\partial \lambda_B}\right)^k_m . \tag{2.7.53}$$

Now we use

$$-\ell\frac{\partial \lambda^i_B}{\partial \ell} = -\beta^p\frac{\partial \lambda^i_B}{\partial g^p} . \tag{2.7.54}$$

$$\begin{aligned}
-\ell\frac{\partial}{\partial \ell}B^k_{ji} &= -\frac{\partial^2}{\partial g^i \partial g^j}\left[\beta^p\frac{\partial \lambda^m_B}{\partial g^p}\right]\left(\frac{\partial g}{\partial \lambda_B}\right)^k_m + B^n_{ij}\frac{\partial}{\partial g^n}\left[\beta^p\frac{\partial \lambda^m_B}{\partial g^p}\right]\left(\frac{\partial g}{\partial \lambda_B}\right)^k_m \\
&= -\beta^k_{,ij} - \beta^p_{,i}B^k_{jp} - \beta^p_{,j}B^k_{ip} - \beta^m B^k_{ij,m} - \beta^m B^n_{ij}B^k_{nm} + B^n_{ij}(\beta^k_{,n} + \beta^p B^k_{np}) \\
&= -\beta^k_{,ij} - \beta^p_{,i}B^k_{jp} - \beta^p_{,j}B^k_{ip} - \beta^m(B^k_{ij,m} + B^n_{ij}B^k_{nm}) + B^n_{ij}\mathcal{D}_n\beta^k .
\end{aligned} \tag{2.7.55}$$

Is the right-hand-side a tensor? Yes it is! In fact, we compute like this:

$$\begin{aligned}
-\beta^k_{,ji} &= -\partial_i\left[\mathcal{D}_j\beta^k - B^k_{jp}\beta^p\right] \\
&= -\mathcal{D}_i\mathcal{D}_j\beta^k - B^n_{ij}\mathcal{D}_n\beta^k + B^k_{im}\mathcal{D}_j\beta^m + \beta^m B^k_{jm,i} + B^k_{jm}\mathcal{D}_i\beta^m - B^k_{jp}B^p_{im}\beta^m \\
-\beta^p_{,i}B^k_{jp} &= -\mathcal{D}_i\beta^m B^k_{jm} + \beta^m B^p_{im}B^k_{jp} \\
-\beta^p_{,j}B^k_{ip} &= -\mathcal{D}_j\beta^m B^k_{im} + \beta^m B^p_{jm}B^k_{ip} ,
\end{aligned} \tag{2.7.56}$$

So that adding these terms to the rest we have

$$\begin{aligned}
-\ell\frac{\partial}{\partial \ell}B^k_{ij} &= \beta^m\left[\partial_i B^k_{jm} - \partial_m B^k_{ji} + B^k_{ip}B^p_{jm} - B^k_{mp}B^p_{ji}\right] - \mathcal{D}_i\mathcal{D}_j\beta^k \\
&= \beta^m F^k_{imj} - \mathcal{D}_i\mathcal{D}_j\beta^k .
\end{aligned} \tag{2.7.57}$$

The c–theorem

In this section we remind the reader of Zamolodchikov's c-theorem that holds in any renormalizable two-dimensional QFT. We present the argument as it is phrased in Polchinski [10].

Rotational invariance implies that the functions

$$\begin{aligned} F &= z^4\langle T(z,\overline{z})T(0)\rangle \ , \\ G &= 4z^3\overline{z}\langle T(z,\overline{z})\Theta(0)\rangle = 4z^3\overline{z}\langle \Theta(z,\overline{z})T(0)\rangle \ , \\ H &= 16z^2\overline{z}^2\langle\Theta(z,\overline{z})\Theta(0)\rangle \end{aligned} \tag{2.7.58}$$

only depend on $r^2 = z\overline{z}$.

Exercise 2.21. On rotationally invariant functions $\mathcal{S} = 2z\partial_z = 2\overline{z}\bar{\partial}_{\overline{z}} = 2\frac{d}{d\log r^2}$. Denote derivatives with respect to $\log r^2$ by a dot. Show that rotational invariance and conservation of the energy-momentum tensor imply the relations

$$4\dot{F} + \dot{G} - 3G = 0 \ , \qquad 4\dot{G} - 4G + \dot{H} - 2H = 0 \ , \tag{2.7.59}$$

and therefore

$$C = 2F - G - \frac{3}{8}H \tag{2.7.60}$$

satisfies

$$r^2\frac{dC}{dr^2} = -\frac{3}{4}H \ . \tag{2.7.61}$$

If we set $r = \ell$, our renormalization scale, and use the relation (2.7.22), we therefore obtain

$$-\ell\frac{\partial C}{\partial \ell} = -24\pi^2\sum_{i,j}\beta^i\beta^j\ell^4\langle\widehat{\mathcal{O}}_i(\ell)\widehat{\mathcal{O}}_j(0)\rangle \ . \tag{2.7.62}$$

Unitarity implies that the Zamolodchikov metric $\ell^4\langle\widehat{\mathcal{O}}_i(\ell)\widehat{\mathcal{O}}_j(0)\rangle$ is positive-definite. Hence, as ℓ increases the C function decreases monotonically along the RG flow, and the decrease terminates if and only if $\beta^i = 0$ for all couplings. In a unitary and compact theory the flow must terminate, and the resulting scale-invariant theory is conformal [9], with central charge given by the critical value of C.

Renormalized CPT

Conformal perturbation theory is a very general method for obtaining renormalized QFTs in the above sense, and at the qualitative level it is remarkably powerful. Conventional perturbation theory around a free field theory is a limit of the more general situation, and in fact a somewhat pathological limit.

In describing the space of QFTs at best we might hope that there is some point $p \in \mathcal{M}$ where a perturbation expansion has a non-zero radius of convergence, with radius given by the distance to the nearest singular point. It is well-known that this hope does not hold when p is a free field theory. On the other hand, this hope is believed to be realized in conformal perturbation theory around a compact unitary CFT. A general argument for this is sketched out in [42], and many exact results suggest this to be the case. However, to the author's knowledge there is no proof of this remarkable conjecture. As the next few paragraphs will indicate, there is probably a good reason for this: our (or at least the author's) understanding of the technical aspects of CPT is still rather limited.

To describe CPT, we must make two aspects of the problem more precise: we must define regulated correlation functions and then give a renormalization prescription. For simplicity we will only discuss correlation functions of the deforming operators $\mathcal{O}_i$.

Regularization

To present the regularized correlators we simply take the formal power-series from (2.7.1) and define the normalized correlation functions as

$$\langle\!\langle \mathcal{O}_{i_1}\cdots\mathcal{O}_{i_s}\rangle\!\rangle = \frac{\langle \mathcal{O}_{i_1}\cdots\mathcal{O}_{i_s}e^{-S}\rangle}{\langle e^{-S}\rangle}\ , \tag{2.7.63}$$

where $S=\int\sum_i \lambda_B^i\mathcal{O}_i$, and the λ_B^i are dimensionful bare couplings.

Next, for every operator with non-zero scaling dimension we define $\mathcal{O}'_i = \mathcal{O}_i - \langle\!\langle\mathcal{O}_i\rangle\!\rangle$, and we formally expand the exponentials to obtain

$$\langle\!\langle \mathcal{O}'_{i_1}\cdots\mathcal{O}'_{i_s}\rangle\!\rangle = \sum_n \frac{(-1)^n}{n!}\lambda_B^{j_1}\cdots\lambda_B^{j_n}\int_{\mathbb{C}^n} d^2z_1\cdots d^2z_n G_{i_1\cdots i_s;j_1\cdots j_n}\ . \tag{2.7.64}$$

Finally, we obtain a well-defined expression by changing the integration domain from $\mathbb{C}^n$ to $\mathbb{D}_{s,n}$, with

$$\mathbb{D}_{s;n} = \{(z_1,\ldots,z_n)\in\mathbb{C}^n\ \ |\ \ |z_{\alpha\beta}|\geq a\ ,\ \ |z_\alpha - w_s|\geq a\}\ . \tag{2.7.65}$$

This explicitly removes all of the UV divergences; we may also need to regulate IR divergences, which may be accomplished by restricting the z_α to $|z_\alpha|<L$. All of the explicit examples we consider will be free of IR divergences, so we will not need to consider this subtlety.

With this set-up in hand we expand the regulated two-point function to second order:

$$\begin{aligned}\langle\!\langle \mathcal{O}'_i(x)\mathcal{O}'_j(0)\rangle\!\rangle_{\text{reg}} &= x^{-2\Delta_i}\delta_{ij} - \lambda_B^k\int_{\mathbb{D}_{2,1}} d^2z\langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k(z)\rangle \\ &\quad + \frac{1}{2}\lambda_B^k\lambda_B^m\int_{\mathbb{D}_{2,2}} d^2z_1d^2z_2\langle\mathcal{O}_i(x)\mathcal{O}_j(0)\mathcal{O}_k(z_1)\mathcal{O}_m(z_2)\rangle_c + O(\lambda_B^3)\ ,\end{aligned} \tag{2.7.66}$$

where the connected four-point function is

$$\langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\mathcal{O}_m\rangle_c = \langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\mathcal{O}_m\rangle - \langle\mathcal{O}_i\mathcal{O}_j\rangle\langle\mathcal{O}_k\mathcal{O}_m\rangle - \langle\mathcal{O}_i\mathcal{O}_k\rangle\langle\mathcal{O}_j\mathcal{O}_m\rangle - \langle\mathcal{O}_i\mathcal{O}_m\rangle\langle\mathcal{O}_j\mathcal{O}_k\rangle\ . \tag{2.7.67}$$

Renormalization of the two-point function

Next, we introduce a renormalization lengthscale ℓ and dimensionless couplings λ^i which will determine the bare couplings as

$$\lambda_B^i = \ell^{-\eta_i}\left[\lambda^i + \tfrac{1}{2}A^i_{2jk}(\ell/a)\lambda^j\lambda^k + \tfrac{1}{6}A^i_{3jkm}(\ell/a)\lambda^j\lambda^k\lambda^m + O(\lambda^4)\right] \ . \tag{2.7.68}$$

Since the bare couplings are independent of the renormalization scale, we can solve for the β function as[18]

$$\beta^i(\lambda) = \left(\frac{\partial\lambda}{\partial\lambda_B}\right)^i_j \ell\frac{\partial\lambda_B^j}{\partial\ell} = -\eta_i\lambda^i + \tfrac{1}{2}B^i_{2jk}\lambda^j\lambda^k + \tfrac{1}{6}B^i_{3jkl}\lambda^j\lambda^k\lambda^m + O(\lambda^3). \tag{2.7.69}$$

Since β should not have an explicit ℓ dependence, we obtain constraints on the coefficients A_2 and A_3 that appear in the expansion of λ_B:

$$\begin{aligned} B^i_{2jk} &= \dot{A}^i_{2jk} - (\eta_i - \eta_j - \eta_k)A^i_{2jk} \ , \\ B^i_{3jkm} &= \dot{A}^i_{3jkm} - (\eta_i - \eta_j - \eta_k - \eta_m)A^i_{3jkm} \\ &\quad - (A^i_{2jn}B^n_{2km} + A^i_{2kn}B^n_{2jm} + A^i_{2mn}B^n_{2jk}) \ , \end{aligned} \tag{2.7.70}$$

where the B_2 and B_3 are ℓ-independent, and $\dot{f} = \ell df/d\ell$.

These equations are easily integrated, but the full solution is a little bit ponderous. So, we will make a simplification of considering deformations by marginal operators, i.e. $\eta_i = 0$ for all i. In that case, we have a simple form for the coefficients:[19]

$$\begin{aligned} A^i_{2jk} &= B^i_{2jk}\log\ell/a + N^i_{2jk} \ , \\ A^i_{3jkm} &= \tfrac{1}{2}(B^i_{2jn}B^n_{2km} + B^i_{2kn}B^n_{2jm} + B^i_{2mn}B^n_{2jk})\log^2\ell/a + \widehat{B}^i_{3jkm}\log\ell/a + N^i_{3jkm} \ , \end{aligned} \tag{2.7.71}$$

where

$$\widehat{B}^i_{3jkm} = B^i_{3kjm} + (N^i_{2jn}B^n_{2km} + N^i_{2kn}B^n_{2jm} + N^i_{2mn}B^n_{2jk}) \ , \tag{2.7.72}$$

and the $N_{2,3}$ are ℓ–independent integration constants. The B and N coefficients in the expansion play different roles. In a renormalizable theory we should be able to choose the B so as to cancel all UV divergences, while the coefficients N will be determined by renormalization conditions on the correlation functions.

This form for the β-function puts strong constraints on the divergences in a renormalizable CPT. To study these constraints we focus on the two-point function of marginal operators.[20] We define the renormalized operators as in the previous section:

$$\widehat{\mathcal{O}}_i = \textstyle\sum_m Z_i^m \mathcal{O}'_m \ . \tag{2.7.73}$$

[18]The expansion coefficients B_2 and B_3 should not be confused with the connection on coupling space discussed in the previous sections.

[19]More generally the solution will consist of power-law terms, as well as log terms associated to "resonance" conditions like $\eta_i - \eta_j - \eta_k = 0$.

[20]In what follows, aside from some side remarks, we will stick to the marginal deformation case so that we can use the simpler (2.7.71) rather than the general solution to (2.7.70).

We then have

$$Z_i^m = \frac{\partial \lambda_B^m}{\partial \lambda^i} = \delta_i^m + A_{2ik}^m \lambda^k + \tfrac{1}{2} A_{3ikl}^m \lambda^k \lambda^l + O(\lambda^3) \ . \tag{2.7.74}$$

Using this in (2.7.66), we obtain

$$\begin{aligned} x^4 \langle\!\langle \widehat{\mathcal{O}}_i(x) \widehat{\mathcal{O}}_j(0) \rangle\!\rangle = \delta_{ij} &+ (A_{2ik}^j + A_{2jk}^i - J_{1ijk}) \lambda^k \\ &+ \big(A_{2ik}^m A_{2jn}^m + \tfrac{1}{2} A_{3ikn}^j + \tfrac{1}{2} A_{3jkn}^i - A_{2ik}^m J_{1mjn} - A_{2jk}^m J_{1imn} \\ &\quad + \tfrac{1}{2} J_{2ijkn} - \tfrac{1}{2} A_{2kn}^m J_{1ijm} \big) \lambda^k \lambda^n + O(\lambda^3) \ , \end{aligned} \tag{2.7.75}$$

where

$$\begin{aligned} J_{1ijk}(x/a) &= x^4 \int_{\mathbb{D}_{2,1}} d^2 z \langle \mathcal{O}_i(x) \mathcal{O}_j(0) \mathcal{O}_k(z) \rangle \ , \\ J_{2ijkm}(x/a) &= x^4 \int_{\mathbb{D}_{2,2}} d^2 z_1 d^2 z_2 \langle \mathcal{O}_i(x) \mathcal{O}_j(0) \mathcal{O}_k(z_1) \mathcal{O}_m(z_2) \rangle_c \ . \end{aligned} \tag{2.7.76}$$

Renormalization at first order

So, let us consider the divergence structure at first order. From above we see that renormalizability requires that

$$J_{1ijk}(x/a) = J_{1ijk}^{\rm div} \log(x/a) + J_{1ijk}^{\rm fin} \ , \tag{2.7.77}$$

where the coefficients $J_{1ijk}^{\rm div}$ and $J_{1ijk}^{\rm fin}$ are both finite in the $a \to 0$ limit. This is also sufficient. By setting

$$B_{2jk}^i = \tfrac{1}{2} (J_{1jik}^{\rm div} + J_{1kij}^{\rm div} - J_{1jki}^{\rm div}) \ , \tag{2.7.78}$$

we will obtain a finite correlation function. We can fix the remaining ambiguity, the coefficient N_2, by the renormalization condition at $x = \ell$

$$\ell^4 \langle \widehat{\mathcal{O}}_i(\ell) \widehat{\mathcal{O}}_j(0) \rangle = \delta_{ij} + O(\lambda^2) \ . \tag{2.7.79}$$

This determines

$$N_{2jk}^i = \tfrac{1}{2} (J_{1jik}^{\rm fin} + J_{1kij}^{\rm fin} - J_{1jki}^{\rm fin}) \ . \tag{2.7.80}$$

Exercise 2.22. Check that both B_{2jk}^i and N_{2jk}^i are completely determined; the former by eliminating the divergence, and the latter by the renormalization condition. The key is to compare the symmetry properties of the coefficients B and N with the symmetry $J_{1ijk} = J_{1jik}$.

Putting all of this together, the renormalized 2-point function is

$$x^4\langle\widehat{\mathcal{O}}_i(x)\widehat{\mathcal{O}}_j(0)\rangle = \delta_{ij} + \textstyle\sum_k \lambda^k J^{\rm div}_{1ijk}\log \ell/x \ + O(\lambda^2). \tag{2.7.81}$$

We now check explicitly that the integral J_{1ijk} does take the desired form—at most a logarithmically divergent term and a finite remainder. This is easy to do since the integrand is fixed by conformal symmetry. Setting $u = z/x$, we have

$$J_{1ijk} = C_{ijk}\int_{\mathbb{D}_{2,1}} d^2u \frac{1}{|u|^2|u-1|^2} \ , \tag{2.7.82}$$

where the regularization removes disks of radius a/x around $u = 0$ and $u = 1$; the integral is IR finite, so these two sources of UV logarithmic divergence are all we need to consider. Subtracting off those contributions, we have

$$J_{1ijk} = 8\pi C_{ijk}\log x/a + C_{ijk}\times \text{finite} \ . \tag{2.7.83}$$

We reproduce the well-known result that non-vanishing C_{ijk} is an obstruction to conformal invariance at first order in conformal perturbation theory.

Inclusion of relevant operators at first order

At first order it is not very difficult to include relevant operators into the story, as was done in the seminal paper of Zamolodchikov [43]. We will leave the details as an exercise and just give a sketch of the derivation. First, we observe that the relevant integrated three-point function takes the form

$$\begin{aligned}\int d^2z \ \langle\mathcal{O}'_i(x)\mathcal{O}'_j(0)\mathcal{O}'_k(z)\rangle &= x^{2-\Delta_i-\Delta_j-\Delta_k}C_{ijk}\mathcal{I}_{ijk} \ , \\ \mathcal{I}_{ijk} &= \int d^2u|u|^{\eta_k+\eta_j-\eta_i-2}|u-1|^{\eta_k+\eta_i-\eta_j-2} \ .\end{aligned} \tag{2.7.84}$$

This converges as long as $\eta_k+\eta_j-\eta_i > 0$ for all k,j,i (that eliminates UV divergences) and $\eta_k < 1$ (which eliminates IR divergences), in which case with our conventions the integral is the familiar result

$$\mathcal{I}_{ijk} = \frac{2\pi\Gamma(a)\Gamma(b)\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)} \ , \qquad 2a = \eta_k+\eta_j-\eta_i \ , \qquad 2b = \eta_k+\eta_i-\eta_j \ . \tag{2.7.85}$$

Note that while $\mathcal{I}_{ijk} = \mathcal{I}_{jik}$, the integral does not have full i,j,k symmetry. The coefficient B^i_{2jk} of the β function is then determined by requiring that the renormalized correlation functions satisfy

$$\ell^4\langle\widehat{\mathcal{O}}_i(\ell)\widehat{\mathcal{O}}_j(0)\rangle = \delta_{ij} + O(\lambda^2) \ . \tag{2.7.86}$$

This fixes

$$A^i_{2jk} = \frac{B^i_{2jk}}{\eta_k+\eta_j-\eta_i} \ , \qquad B^i_{2jk} = -\frac{1}{2}(\eta_j-\eta_i-\eta_k)C_{ijk}\left(\mathcal{I}_{ijk}+\mathcal{I}_{kji}-\mathcal{I}_{kij}\right) \ , \tag{2.7.87}$$

so that the renormalized correlation function becomes

$$\ell^4\langle\widehat{\mathcal{O}}_i(x)\widehat{\mathcal{O}}_j(0)\rangle_{\rm ren} - \left(\frac{\ell}{x}\right)^{2\Delta_i}\delta_{ij} = \frac{1}{2}\sum_k \lambda^k C_{ijk}\left\{\left(\frac{\ell}{x}\right)^{2\Delta_j}(\mathcal{I}^{jki}-\mathcal{I}^{kij}+\mathcal{I}^{ijk}) + \left(\frac{\ell}{x}\right)^{2\Delta_i}(\mathcal{I}^{ijk}-\mathcal{I}^{kji}+\mathcal{I}^{kij}) - \left(\frac{\ell}{x}\right)^{\Delta_i+\Delta_j+\Delta_k-2}\mathcal{I}^{ijk}\right\} + O(\lambda^2)\ , \tag{2.7.88}$$

while the β function is given by

$$\beta^j = -\eta_j\lambda^j - \frac{\eta_j-\eta_i-\eta_k}{4}C_{ijk}\left[\mathcal{I}^{ijk}+\mathcal{I}^{kji}-\mathcal{I}^{kij}\right]\lambda^i\lambda^k + O(\lambda^3)\ . \tag{2.7.89}$$

This form has a number of important lessons. First, as emphasized in [43], when the η can be made uniformly small, such as in a large level limit of minimal models, this can be used to find non-trivial fixed points at small λ. Second, we see that although C_{ijk} is symmetric in i, j, k, the remaining terms in the cofficient do not have full symmetry, and thus, the β function is not in general the gradient of a scalar function. [21] Indeed, there is no reason that the β function should take a gradient form in general; a nice discussion of this, as well as a modified gradient formula derived in significant generality may be found in [33].

Renormalization at second order

We now return to the case of purely marginal operators but examine renormalization at second order. Taking a look at the second order term in (2.7.75), we observe that all of the underlined terms are already determined by our first order analysis:

$$\Big(\underline{A^m_{2ik}A^m_{2jn}} + \tfrac{1}{2}A^j_{3ikn} + \tfrac{1}{2}A^i_{3jkn}\underline{-A^m_{2ik}J_{1mjn} - A^m_{2jk}J_{1imn}} + \tfrac{1}{2}J_{2ijkn}\underline{-\tfrac{1}{2}A^m_{2kn}J_{1ijm}}\Big)\lambda^k\lambda^n\ . \tag{2.7.90}$$

At this point it is already clear that in a renormalizable theory we will have

$$\begin{aligned} J_{2ijkn} &= J^{\rm div,2}_{2ijkn}\log^2 x/a + J^{\rm div,1}_{2ijkn}\log x/a + J^{\rm fin}_{2ijkn} \\ &= J^{\rm div,2}_{2ijkn}\log^2 x/\ell + J^{\rm div,1}_{2ijkn}\log x/\ell + J^{\rm fin}_{2ijkn} \\ &\quad + J^{\rm div,2}_{2ijkn}\log^2 \ell/a + 2J^{\rm div,2}_{2ijkn}\log x/\ell\log\ell/a + J^{\rm div,1}_{2ijkn}\log\ell/a\ . \end{aligned} \tag{2.7.91}$$

Moreover, from (2.7.71) we can also see that all of the "non-local" $\log^2\ell/a$ and $\log x/\ell\log\ell/a$ terms that arise from J_2 can only be cancelled by coefficients that are already fixed at first

[21] In the small η expansion the β function is a gradient of a scalar function to $O(\lambda^3)$.

order.[22] On the other hand, the $\log \ell/a$ term can be potentially canceled by an appropriate choice of the coefficient B^i_{3jkm}.

The integral of the four-point function is more difficult to analyze in full generality. While conformal invariance does determine it in principle, in practice we rarely have enough information about the theory to get an expression. We expect that locality of the OPE should still be sufficient to at least identify and characterize the divergences, and thus perhaps prove renormalizability of CPT at this order. However, the author is not aware of any detailed arguments that effect this and considers it an important open question in the field; once it has a clear answer, we will have a much better understanding of CPT!

Let us end by pointing out that even in the simplest situation, where J_1 is finite, so that $B_2 = 0$, we will need additional information on the four-point function to show renormalizability. In this case, to produce a finite correlation function we will need to solve

$$-J^{\text{div},1}_{2ijkn} = \tfrac{1}{2}A^j_{3ikn} + \tfrac{1}{2}A^i_{3jkn} \ . \tag{2.7.92}$$

The left-hand side is of course symmetric in i, j as well as k, n, but it does not a priori have any additional symmetries. On the other hand, A^i_{3jkn} is symmetric in j, k, n, and there is potential obstruction to finding a solution. As we will see in the next section, this obstruction is very familiar: it is exactly the same as the obstruction to finding coordinates in which a Riemannian metric is flat — i.e. a non-vanishing Riemann tensor. So, to show renormalizability of the theory, we would need to demonstrate that the divergent terms that arise in the $a \to 0$ limit of the regulated integrals are flat in an appropriate sense. Making that statement precise would be a very useful step in developing conformal perturbation theory and perhaps proving its renormalizability under a suitable set of assumptions.

The Zamolodchikov metric on the moduli space

We mentioned a few times that a family of unitary CFTs with exactly marginal deformations is naturally equipped with a Riemannian metric. It is very easy to see how this comes about. Suppose we can make sense of CPT as far as regularization and renormalization goes, and we find that a family of perturbations preserves conformal invariance. In that case, the two-point function of the exactly marginal operators takes the form

$$g_{ij}(\lambda) = x^4 \langle \widehat{\mathcal{O}}_i(x)\widehat{\mathcal{O}}_j(0)\rangle_{\text{ren}} = \delta_{ij} + \textstyle\sum_k \kappa_{1ij,k}\lambda^k + \frac{1}{2}\sum_{k,m}\kappa_{2ij,km}\lambda^k\lambda^m + O(\lambda^3) \ . \tag{2.7.93}$$

The coefficients κ are not determined by conformal invariance, except to have the obvious symmetries: $\kappa_{ij,\ldots} = \kappa_{ji,\ldots}$, as well as full symmetry in the indices that follow the comma.

Of course at this point we still have a freedom of modifying our coordinates to $\lambda^i = \lambda^i(\lambda')$, which in a pertrubative expansion takes the form

$$\lambda^i = \lambda'^i + \tfrac{1}{2}\textstyle\sum_{j,k} b^i_{1jk}\lambda'^j\lambda'^k + \frac{1}{3!}\sum_{j,k,m} b^i_{2jkm}\lambda'^j\lambda'^k\lambda'^m + O(\lambda'^4) \ . \tag{2.7.94}$$

[22]This should remind the reader of overlapping divergences in standard perturbation theory computations around a free field theory.

Given our assumptions of a geometric renormalization, $g = \sum_{i,j} g_{ij} d\lambda^i d\lambda^j$ is a tensor, and it is an easy matter to write down its form in the λ' coordinate system. As the following exercise shows, we can always follow Riemann and find a coordinate system such that

$$g'_{ij} = \delta_{ij} + \tfrac{1}{3} \sum_{k,m} R_{ikjm} \lambda'^k \lambda'^m + O(\lambda'^3) \ , \tag{2.7.95}$$

where R is the Riemann tensor associated to the metric g. Of course it is not possible to set R to zero by a choice of coordinates.

Exercise 2.23. In this exercise we remind the reader of the Riemann normal coordinates. Show that at it is always possible to set $\kappa'_{1ij,k}$ to zero by a choice of coordinates by setting

$$b^i_{1jk} = -\tfrac{1}{2} \left(\kappa_{ji,k} + \kappa_{ki,j} - \kappa_{jk,i} \right) \ , \tag{2.7.96}$$

and, moreover, this fixes the coefficients b^i_{1jk} in the coordinate transformation (think about the symmetries of the coefficients to show this last point). Thus, we can, without loss of generality, set $\kappa_{1ij,k} = 0$ and $b^i_{1jk} = 0$ in the expressions above. Next, show that b^i_{2jkm} can be chosen so that

$$\kappa'_{ij,km} = \tfrac{1}{3} \left(\kappa_{2ij,km} + \kappa_{2km,ij} - \tfrac{1}{2} \left[\kappa_{2ik,jm} + \kappa_{2jm,ik} + \kappa_{2ik,jm} \right] \right) \ , \tag{2.7.97}$$

and, moreover, this fixes the coefficient b^i_{2jkm}, so that $\kappa'_{ij,km}$ is a coordinate-independent quantity. Show that it is given by the Riemann tensor as advertised above.[23]

As the exercise shows, to second order in CPT the renormalized correlation function can be cast in terms of a Riemann normal coordinate expansion of a Riemannian metric. This choice completely fixes any left-over ambiguity in the renormalization scheme. Indeed, we could continue to higher orders to find a similar structure: the coordinate freedom would be eliminated, and the "left-over" terms could be taken to be the higher order terms of the Riemann normal expansion, i.e. terms involving covariant derivatives of the Riemann tensor. Of course it is certainly possible to choose other ways to fix the renormalization scheme, which simply corresponds to different choices of coordinates on the space of exactly marginal couplings.

It should be borne in mind that once we assume that we can find the kind of geometric regularization/renormalization scheme that we advertised above, then the conclusions about the existence of the Zamolodchikov metric are more or less built-in from the start: the two-point function must transform as a tensor, and its ambiguities are exactly those of a choice of coordinates. The story becomes more interesting if we can compare the computations in CPT to some other way we might have of obtaining a metric on the moduli space. The following exercise, discussed in [38], is an interesting illustration of this story.

[23] Our conventions are to take the Christoffel symbol as $\Gamma^i_{jk} = \frac{1}{2} \sum_m g^{im} \left(g_{jm,k} + g_{km,j} - g_{jk,m} \right)$ and the Riemann tensor $R_{ijk}{}^m = \Gamma^m_{ik,j} - \Gamma^m_{jk,i} + \sum_n \left[\Gamma^n_{ik} \Gamma^m_{nj} - \Gamma^n_{jk} \Gamma^m_{ni} \right]$. More details are provided in appendix B.2.

Exercise 2.24. Consider a CFT with marginal operators $\mathcal{O}_i = J_i\overline{J}$, where J_i, $i = 1, \ldots, d$ are left-moving U(1) currents and $\overline{J}$ is a single right-moving U(1) current. This theory is an asymmetric torus and has moduli space (up to a discrete quotient) given by

$$\mathbb{H}^d = \text{Gr}(d,1) = \frac{\text{SO}(d,1)}{\text{SO}(d)\times\text{SO}(1)} \ . \tag{2.7.98}$$

Use CPT to compute the second order correction to the moduli space metric and verify that it matches the expectation of a constant negative curvature metric on the hyperbolic space $\mathbb{H}^d$. In this case the integrals involved are fairly simple improper ones that do not need an explicit regularization. The following identities (which the reader is encouraged to verify by integration by parts) can be used to compute all of the necessary integrals:

$$\mathcal{I}_1 = \int \frac{d^2z_3 d^2z_4}{\overline{z}_{13}^2\overline{z}_{24}^2 z_{34}^2} = \frac{(2\pi)^2}{\overline{z}_{12}^2} \ , \qquad \mathcal{I}_2 = \int \frac{d^2z_3 d^2z_4}{z_{13}^2 z_{24}^2 \overline{z}_{14}^2\overline{z}_{23}^2} = 0 \ , \qquad z_{12}\neq 0 \ . \tag{2.7.99}$$

%%%Maybe add contact terms and relationship of $g_{ij,k}$ to $\langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle$.

A few more formal comments

A few more comments regarding this formal tale are probably useful. Perhaps the most important point to keep in mind is that while such a formal point of view may clarify the structure of difficult problems in QFT, it certainly does not solve them. For instance, we certainly expect that there will be strong coupling singularities in the coupling space $\mathcal{M}$ that cannot be described in terms of a perturbation theory around a smooth point $p\in\mathcal{M}$. In some cases the singularity may have an interpretation of a coordinate singularity, but in others it may be genuine, indicating that the theory exhibits a singular behavior: renormalized correlation functions diverge, and the divergence cannot be eliminated by a change of coordinates.

2.8 CPT for (0,2) CFTs

Having plumbed the depths of the author's ignorance of conformal perturbation theory in general, we can now turn to a more specific application to supersymmetric perturbations of superconformal theories.[24]

Supersymmetric deformations at first order

In making a CPT expansion in a non-supersymmetric theory we considered perturbations defined by

$$S[\lambda] = \int d^2z \sum_i \lambda^i \mathcal{O}_i(z,\overline{z}) \ , \tag{2.8.1}$$

[24]This section follows the discussion given in [44], which is in turn based on ideas to be found in [45], as well as the much earlier [46].

where $\mathcal{O}_i$ are spin 0 operators; the latter restriction ensures that the first order deformation preserves Lorentz invariance; moreover, we saw above that we can easily find Lorentz-invariant regularization schemes, so that CPT is assured to remain Lorentz-invariant.

In a supersymmetric theory it is natural to consider deformations that also preserve some amount of supersymmetry. We will now show that if we wish to preserve the full (0,2) supersymmetry and reality of the action, then the deformations take the form[25]

$$\mathcal{O}(z,\bar{z}) = \Big[\{G^-_{-1/2}, \mathcal{U}(z,\bar{z})\} + \text{h.c.}\Big] + \{G^+_{-1/2}, [G^-_{-1/2}, \mathcal{K}(z,\bar{z})]\} \ , \tag{2.8.2}$$

where $\mathcal{U}$ is a fermionic chiral primary operator with

$$\bar{h}_{\mathcal{U}} = 1/2 + h_{\mathcal{U}} \ , \qquad\qquad h_{\mathcal{U}} = q_{\mathcal{U}}/2 \ , \tag{2.8.3}$$

and $\mathcal{K}$ is a real operator with $\bar{h}_{\mathcal{K}} - h_{\mathcal{K}} = 1$. Our proof applies to any compact and unitary CFT with N=2 supersymmetry.

To prove the claim it is convenient to use the state-operator correspondence and work a state $|\mathcal{O}\rangle$. The first simplification is that $|\mathcal{O}\rangle$ can be taken to be Virasoro quasi-primary; any other term will be a total derivative and will not contribute to the integrated action; of course $|\mathcal{O}\rangle$ will have 0 spin to preserve Lorentz invariance and will be bosonic.

Next, for the deformation to be supersymmetric we will need

$$G^{\mp}_{-1/2}|\mathcal{O}\rangle = L_{-1}|\mathcal{M}^{\mp}\rangle \tag{2.8.4}$$

for some states $|\mathcal{M}^{\mp}\rangle$. Applying $G^{\pm}_{-1/2}$ to both sides of the equation and using the N=2 algebra leads to

$$L_{-1}\Big[|\mathcal{O}\rangle - G^+_{-1/2}|\mathcal{M}^-\rangle - G^-_{-1/2}|\mathcal{M}^+\rangle\Big] = 0 \ . \tag{2.8.5}$$

But that in turn means that up to a constant multiple of the identity operator, which would lead to a trivial deformation of the theory, we can write

$$\begin{aligned} |\mathcal{O}\rangle &= G^-_{-1/2}|\mathcal{M}^+\rangle + G^+_{-1/2}|\mathcal{M}^-\rangle \\ &= G^-_{-1/2}|\mathcal{U}\rangle + G^+_{-1/2}|\mathcal{V}\rangle + \left[G^+_{-1/2}G^-_{-1/2} - \left(1 + \frac{q_{\mathcal{K}}}{2h_{\mathcal{K}}}\right)L_{-1}\right]|\mathcal{K}\rangle \ , \end{aligned} \tag{2.8.6}$$

where $|\mathcal{U}\rangle$, $|\mathcal{V}\rangle$, and $|\mathcal{K}\rangle$ are all quasi-primary with respect to the N=2 superconformal algebra, i.e. annihilated by the lowering modes of the global N=2 algebra, L_1, and $G^{\pm}_{1/2}$. The slightly mysterious linear combination of operators in the last term is fixed by the requirement $L_1|\mathcal{O}\rangle = 0$. At this point we see that Lorentz invariance requires $\bar{h}_{\mathcal{K}} = h_{\mathcal{K}} + 1$ and supersymmetry does not further constrain the operator $\mathcal{K}$.

[25]Note that in this section we still keep the N=2 algebra on the holomorphic side of the worldsheet, i.e. we are really speaking of a (2,0) theory; we will switch it to the anti-holomorphic when it will be more convenient to do so.

Because it is the integral of $\mathcal{O}$ that appears in the deformation, the only remaining constraints from supersymmetry are

$$G^+_{-1/2}G^-_{-1/2}|\mathcal{U}\rangle = L_{-1}|X\rangle \ , \qquad G^-_{-1/2}G^+_{-1/2}|\mathcal{V}\rangle = L_{-1}|Y\rangle \tag{2.8.7}$$

for some states $|X\rangle$ and $|Y\rangle$.

We will now show that the only solution to these conditions is to take $|\mathcal{U}\rangle$ to be chiral primary and $|\mathcal{V}\rangle$ to be anti-chiral primary. It suffices to show the first statement, as the second follows simply by exchanging $|\mathcal{U}\rangle$ with $|\mathcal{V}\rangle$ and G^+ with G^-. Without loss of generality we decompose

$$|X\rangle = a|\mathcal{U}\rangle + |\chi\rangle \ , \tag{2.8.8}$$

where a is a constant and $|\chi\rangle$ is orthogonal to $|\mathcal{U}\rangle$. Using this decomposition we obtain the condition

$$\left[G^+_{-1/2}G^-_{-1/2} - aL_{-1}\right]|\mathcal{U}\rangle = L_{-1}|\chi\rangle \ . \tag{2.8.9}$$

Observe that

$$\langle\mathcal{U}|L_1L_{-1}|\chi\rangle = \langle\mathcal{U}|2L_0|\chi\rangle + \langle\mathcal{U}|L_{-1}L_1|\chi\rangle = 0 \ . \tag{2.8.10}$$

The first term vanishes because of the orthogonality assumption, and the second is zero because $|\mathcal{U}\rangle$ is quasi-primary. Hence, we can apply $\langle\mathcal{U}|L_1$ to both sides of (2.8.9) to find

$$\langle\mathcal{U}|L_1\left[G^+_{-1/2}G^-_{-1/2} - aL_{-1}\right]|\mathcal{U}\rangle = 0 \qquad \Longleftrightarrow \qquad a = 1 + \frac{q_\mathcal{U}}{2h_\mathcal{U}} \ . \tag{2.8.11}$$

On the other hand, applying $\langle\chi|L_1$ to both sides of (2.8.9), we find $\|L_{-1}|\chi\rangle\|^2 = 0$. Hence, we now have

$$\left[G^+_{-1/2}G^-_{-1/2} - \left(1 + \frac{q_\mathcal{U}}{2h_\mathcal{U}}\right)L_{-1}\right]|\mathcal{U}\rangle = 0 \ . \tag{2.8.12}$$

We now apply $G^-_{1/2}$, and the N=2 algebra yields

$$\left[2L_0 - J_0 + \left(1 + \frac{q_\mathcal{U}}{2h_\mathcal{U}}\right)\right]G^-_{-1/2}|\mathcal{U}\rangle = 0 \ . \tag{2.8.13}$$

So, either $G^-_{-1/2}|\mathcal{U}\rangle = 0$, or

$$2\left(h_\mathcal{U} + \frac{1}{2}\right) - (q-1) - \left(1 + \frac{q_\mathcal{U}}{2h_\mathcal{U}}\right) = 0 \ . \tag{2.8.14}$$

The former possibility leads to a trivial deformation; the latter leads is equivalent to

$$(1 + 2h_\mathcal{U})\left(1 - \frac{q_\mathcal{U}}{2h_\mathcal{U}}\right) = 0 \ , \tag{2.8.15}$$

and the only solution consistent with unitarity is $h_{\mathcal{U}} = q_{\mathcal{U}}/2$. In other words, $|\mathcal{U}\rangle$ is a chiral primary operator, as was to be shown. Lorentz invariance then requires $\overline{h}_{\mathcal{U}} = h_{\mathcal{U}} + 1/2$.

An exactly parallel treatment shows that $|\mathcal{V}\rangle$ must be an anti-chiral primary state; of course then reality of the deformation requires that the two terms should be conjugate. Finally, returning to the corresponding operators, it follows that we have now established the claim (2.8.2).

Furthermore, we see that the deformations have weights and R-charges

$$\begin{array}{ccccc} \text{operator} & h & \overline{h} & q & \Delta \\ \{G^-_{-1/2}, \mathcal{U}\} & h_{\mathcal{U}} + \frac{1}{2} & h_{\mathcal{U}} + \frac{1}{2} & q_{\mathcal{U}} - 1 & 1 + q_{\mathcal{U}} \\ \{G^+_{-1/2}, [G^-_{-1/2}, \mathcal{K}(z,\overline{z})]\} & h_{\mathcal{K}} + 1 & h_{\mathcal{K}} + 1 & q_{\mathcal{K}} & 2 + 2h_{\mathcal{K}} \end{array} \tag{2.8.16}$$

In particular, there are no marginal or relevant deformations of the $\mathcal{K}$ type: the borderline case is $h_{\mathcal{K}} = 0$, i.e. $\mathcal{K}$ is an anti-holomorphic current, but that is a trivial deformation since the corresponding operator would then be annihilated by the supercharges. On the other hand, we can have relevant and marginal $\mathcal{U}$ operators; the former requires $q_{\mathcal{U}} < 1$, while the latter requires $q_{\mathcal{U}} = 1$. We see that, as expected, marginal deformations will preserve the R-symmetry at first order in the deformation.

Exercise 2.25. Show that a (0,2) marginal deformation of a (0,4) CFT is necessarily (0,4) marginal. Hint: use the SU(2) Kac-Moody algebra.

Basic constraints on RG flow

In this section we consider flows due to a relevant supersymmetric deformation of a known UV fixed point to an IR fixed point. Such flows are provide what is certainly the simplest route to the largest set of non-trivial (0,2) SCFTs. We will take up the exploration of this zoo of theories in the following chapters, but for now we will simply point out a few obvious facts that can guide any nascent (0,2) explorer.

Consequences of unitarity bound

Having identified the marginal operators, we can discuss their fate under a finite deformation. As we discussed above, in a general CFT this is a difficult question. In a supersymmetric theory we have additional tools at our disposal. For instance, *if* we have a marginal coupling λ such that the $\mathrm{U}(1)_{\mathrm{R}}$ charges are λ–independent, it is very easy to see that a marginal supersymmetric operator $\mathcal{O}$ at $\lambda = 0$ can cease to be marginal for $\lambda \neq 0$ in just one of two ways:

1. "F–term" : in this case $\mathcal{O}$ is lifted together with an anti-chiral primary $\overline{\mathcal{F}}$ with $h = 1$, $\overline{h} = -\overline{q}/2 = 1$, for $\lambda \neq 0$ the two chiral-primary operators combine into a long multiplet with lowest component having $\overline{q} = 1$ and weight $\overline{h} = 1/2 + \epsilon(\lambda)$.

2. "D–term" : in this case $\mathcal{O}$ is lifted together with a left-moving KM current J: for $\lambda \neq 0$ J and $\mathcal{O}$ now combine into a long multiplet with lowest component having $\overline{q} = 1$ and weight $\overline{h} = \epsilon(\lambda)$.

This was discussed many years ago in [46] . The crucial assumption in this discussion is the λ– independence of $\mathrm{U}(1)_{\mathrm{R}}$ charges. It is not known to the author what are the general conditions on a (0,2) SCFT where this condition holds.

It is tempting to translate the into a more general statement about conformal perturbation theory. For instance, we might expect that quite generally the unitarity bound $\overline{h} \geq \overline{q}/2$ will continue to hold under a general marginal deformation. Together with a statement of $\mathrm{U}(1)_{\mathrm{R}}$–charge invariance this would imply that a marginal deformation can at worst become marginally irrelevant. This seems sensible, and there are theories where the $\mathrm{U}(1)_{\mathrm{R}}$ charges are λ–independent, but the author is not aware of a proof that this is always the case. It would be very useful to have a more precise statement of the unitarity bound in the language of supersymmetric conformal perturbation theory. The basic technical hurdle to overcome is that, unlike the case of chiral operators in $N = 1$ $d = 4$ superconformal theories, in general (0,2) theories chiral primary operators can and do have singularities in the OPE. The singularities will be holomorphic, but that is not enough (as far as the author is aware) to formulate a formal non-renormalization argument of the sort used in [45].

Consequences of chirality

Consider a relevant deformation of (0,2) UV fixed point with central charges $c_{\mathrm{uv}}, \overline{c}_{\mathrm{uv}}$ that flows to an IR fixed point with $c_{\mathrm{ir}}, \overline{c}_{\mathrm{ir}}$. As we discussed above, the local diffeomorphism anomaly in the UV theory is proportional to the difference $c_{\mathrm{uv}} - \overline{c}_{\mathrm{uv}}$. This anomaly must also be present in the IR theory, whence we learn that

$$c_{\mathrm{uv}} - \overline{c}_{\mathrm{uv}} = c_{\mathrm{ir}} - \overline{c}_{\mathrm{ir}} \ . \tag{2.8.17}$$

Any time $c_{\mathrm{uv}} \neq c_{\mathrm{ir}}$ we are guaranteed to have a non-trivial IR fixed point of the flow!

Indeed, for compact and unitary theories we have a stronger statement in Zamolodchikov's theorem. First, of course $c_{\mathrm{uv}} > c_{\mathrm{ir}}$. Moreover, the difference of Zamolodchikov's C-function and its anti-holomorphic twin $\overline{C}$ is an RG invariant:

$$-\ell \frac{\partial}{\partial \ell} \left[C - \overline{C} \right] = 0 \ . \tag{2.8.18}$$

Exercise 2.26. Review Zamolodchikov's argument given above and prove this assertion.

A similar, in fact simpler, constraint also exists for any conserved current. Let $J, \overline{J}$ be the components of a conserved current, i.e. we have an operator equation in the renormalized theory

$$\bar{\partial} J + \partial \overline{J} = 0 \ . \tag{2.8.19}$$

Consider then the following rotationally invariant correlation functions:

$$\begin{aligned} K &= z^2\langle J(z,\bar{z})J(0)\rangle \ , \qquad M = z\bar{z}\langle J(z,\bar{z})\overline{J}(0)\rangle = z\bar{z}\langle \overline{J}(z,\bar{z})J(0)\rangle \ , \\ \overline{K} &= \bar{z}^2\langle \overline{J}(z,\bar{z})\overline{J}(0)\rangle \ . \end{aligned} \tag{2.8.20}$$

Conservation and rotational invariance imply

$$\dot{K} + \dot{M} = M \ , \qquad \dot{\overline{K}} + \dot{M} = M \ , \tag{2.8.21}$$

where $\dot{K} = r\frac{\partial}{\partial r}K$ and $r^2 = z\bar{z}$, and as a result we have another RG-invariant quantity:

$$\ell\frac{\partial}{\partial \ell}\left[K - \overline{K}\right] = 0 \ . \tag{2.8.22}$$

At the conformal points, where (in unitary and compact theories) K and $\overline{K}$ reduce to the levels of, respectively, J and $\overline{J}$, while $M = 0$, so that RG-invariance implies

$$k_{\rm uv} - \bar{k}_{\rm uv} = k_{\rm ir} - \bar{k}_{\rm ir} \ . \tag{2.8.23}$$

The statement also follows from 't Hooft anomaly matching applied to the two-dimensional theory.

It follows that in any RG flow to a compact unitary CFT, if $k_{\rm uv} - \bar{k}_{\rm uv} > 0$, then the IR fixed point must have a left-moving chiral symmetry; similarly, if $k_{\rm uv} - \bar{k}_{\rm ir} < 0$, then the IR fixed point must have a right-moving chiral symmetry.

Finding the R-symmetry in the IR

Note that in this section we will revert to the convention that supersymmetry is on the right, i.e. anti-holomorphic side of the world-sheet.

Suppose there is a (0,2) supersymmetric flow with an indecomposable supercurrent multiplet from a UV fixed point to some compact unitary (0,2) SCFT. In such a situation we may hope to identify at least some of the SCFT spectrum with operators constructed from the UV fields. For instance, as we will discuss in greater length below, we might expect to reliably study the spectrum of chiral operators that correspond to $\overline{\boldsymbol{Q}}_+$-cohomology. A crucial element in any such analysis is the proper identification of the IR R-symmetry. There are two possibilities, and each realized in (0,2) dynamics. It may be that the IR R-symmetry current is obtained as a linear combination of the UV R-symmetry and UV global U(1) symmetries. In that case the IR N=2 algebra is indecomposable, and the technique of *c-extremization* [47] fixes the precise linear combination. The alternative is that the IR R-symmetry mixes with an accidental IR symmetry. Not much can be said in general about this situation, but we will present some well-motivated conjectures in some examples below.

We now examine the first possibility in more detail and review the elegant method of c-extremization. Assume that we are given a UV theory with an indecomposable supercurrent multiplet $\overline{\mathcal{S}}$ and conserved abelian currents[26]

$$\mathcal{J}^0_{\rm uv} = J^0_{\rm uv}\frac{\partial}{\partial z} + \overline{J}^0_{\rm uv}\frac{\partial}{\partial \overline{z}} ~, \qquad\qquad \mathcal{J}^\alpha_{\rm uv} = J^\alpha_{\rm uv}\frac{\partial}{\partial z} + \overline{J}^\alpha_{\rm uv}\frac{\partial}{\partial \overline{z}} ~, \tag{2.8.24}$$

where $\mathcal{J}_0$ is the UV R-symmetry and $\alpha = 1, \ldots, k$ label the remaining global symmetries. We assume that these symmetries are preserved by the RG flow, and its IR limit is a unitary compact (0,2) SCFT with an indecomposable supercurrent multiplet whose lowest component, the IR R-current $\mathcal{J}_{\rm ir} = \overline{J}_{\rm ir}\frac{\partial}{\partial\overline{z}}$ is a linear combination of the abelian UV symmetries. More precisely, there is some choice of k real parameters t_α such that the current

$$\mathcal{J}^t = \mathcal{J}^0_{\rm ir} + \sum_{\alpha=1}^{k} t_\alpha \mathcal{J}^\alpha_{\rm uv} \tag{2.8.25}$$

flows to $\mathcal{J}_{\rm ir}$. We will now show that 't Hooft anomaly matching uniquely determines the parameters t^α and therefore fixes the presentation of the IR R-symmetry in terms of the UV degrees of freedom.

To apply 't Hooft anomaly matching we couple the currents $\mathcal{J}^t_{\rm uv}$ and $\mathcal{J}^\alpha_{\rm uv}$ to background gauge fields $\mathcal{A}_t$ and $\mathcal{A}_\alpha$.[27] As we saw in section 2.6, the effective action will in general transform under gauge transformations $\mathcal{A}_t \to \mathcal{A}_t - df_t$ and $\mathcal{A}_\alpha \to \mathcal{A}_\alpha - df_\alpha$. We will choose a regularization scheme where the anomaly takes its symmetric form, so that

$$\delta_f W[\mathcal{A}] = -\frac{i}{4\pi}\int \left\{ -\frac{\overline{C}(t)}{3} f^t \mathcal{F}^t + \sum_\alpha L^\alpha(t)(f_t\mathcal{F}_\alpha + f_\alpha \mathcal{F}_t) + \sum_{\alpha,\beta} M^{\alpha\beta} f_\alpha \mathcal{F}_\beta \right\} ~. \tag{2.8.26}$$

Here the $\mathcal{F}$ are the gauge field strengths, while the coefficients $\overline{C}(t)$, $L(t)$ and $M^{\beta\alpha} = M^{\alpha\beta}$ are polynomial in t and determined by the UV current two-point functions; namely, we have

$$\begin{aligned} a^{00} &= \lim_{z\to 0}\left[z^2 \langle J^0_{\rm uv}(z,\overline{z}) J^0_{\rm uv}(0)\rangle - \overline{z}^2 \langle \overline{J}^0_{\rm uv}(z,\overline{z}) \overline{J}^0_{\rm uv}(0)\rangle \right] ~, \\ a^{0\beta} &= \lim_{z\to 0}\left[z^2 \langle J^0_{\rm uv}(z,\overline{z}) J^\beta_{\rm uv}(0)\rangle - \overline{z}^2 \langle \overline{J}^0_{\rm uv}(z,\overline{z}) \overline{J}^\beta_{\rm uv}(0)\rangle \right] ~, \\ M^{\alpha\beta} &= \lim_{z\to 0}\left[z^2 \langle J^\alpha_{\rm uv}(z,\overline{z}) J^\beta_{\rm uv}(0)\rangle - \overline{z}^2 \langle \overline{J}^\alpha_{\rm uv}(z,\overline{z}) \overline{J}^\beta_{\rm uv}(0)\rangle \right] ~, \end{aligned} \tag{2.8.27}$$

and

$$\begin{aligned} -\frac{1}{3}\overline{C}(t) &= a^{00} + 2\sum_\beta t_\beta a^{0\beta} + \sum_{\alpha,\beta} t_\alpha t_\beta M^{\alpha\beta} ~, \\ L^\beta(t) &= a^{0\beta} + \sum_\gamma t_\gamma M^{\gamma\beta} ~. \end{aligned} \tag{2.8.28}$$

[26]We restrict attention to abelian currents, since non-abelian currents cannot mix with the R-symmetry.
[27]Please note that $\mathcal{A}_t$ has no t dependence; it is simply the background gauge field that couples to $\mathcal{J}^t$.

The key point is that this full anomaly must be reproduced by IR degrees of freedom. Let us now discuss how this might happen. As a first step, we derive some constraints on the symmetric matrix M.

The $k \times k$ symmetric matrix M can be diagonalized and decomposed according to the sign of the eigenvalues: $M = M^- \oplus M^0 \oplus M^+$. With our assumptions M^- must be absent: if it were not, then the IR theory necessarily has an additional anti-holomorphic current that will reproduce the corresponding term in the anomaly. However, an additional anti-holomorphic current means that the IR supercurrent multiplet is reducible: the additional current must either be the bottom component of a separate supercurrent multiplet, or it resides in an N=2 SKM multiplet, in which case the IR supercurrent multiplet is decomposable by the N=2 super-Sugawara construction.

A non-trivial M^0 means that there is some UV current with zero anomaly. Such a current must choose between two fates in the IR: it must either decouple, or it flows to *two* conserved IR currents with equal level: one holomorphic and one anti-holomorphic. The second fate means again that we have an accidental symmetry; the first fate means that we can ignore every current that corresponds to an element of $\ker M$.

So, with our assumptions we conclude that we can restrict to currents $\mathcal{J}^\alpha_{\text{uv}}$ that are orthogonal to $\ker M$; in order to keep notation simple, we will from now on assume that the index α runs over just such currents. Furthermore, we see that the restricted M must be positive definite; otherwise we cannot hope to match the anomaly. Thus, every $\mathcal{J}^{\text{uv}}_\alpha$ must flow to a holomorphic current $\mathcal{J}^{\text{ir}}_\alpha = J^{\text{ir}}_\alpha \frac{\partial}{\partial z}$. Therefore, the gauge variation of the effective action $W[\mathcal{A}]$ in terms of the IR degrees of freedom must take the form

$$\delta_f W[\mathcal{A}] = -\frac{i}{4\pi} \int \left\{ -\frac{\overline{c}^{\text{ir}}}{3} f^t \mathcal{F}^t + \sum_{\alpha,\beta} f^\alpha \mathcal{F}^\beta M^{\alpha\beta} \right\} . \tag{2.8.29}$$

Comparing (2.8.29) and (2.8.26), we see that we must choose the values $t_\alpha = t^{\text{ir}}_\alpha$ so that $L(t^{\text{ir}}) = 0$, in which case $\overline{C}(t^{\text{ir}}) = \overline{c}_{\text{ir}}$. Since M is by assumption invertible, $t_\alpha = t^{\text{ir}}_\alpha$ is the unique solution to $L(t) = 0$. This is nicely summarized by the statement that "the IR R-symmetry maximizes the trial central charge $\overline{C}(t)$."[28]

Thus, with the assumptions we made, all we need to obtain the IR R-symmetry is the set of current two-point function coefficients. These coefficients are very familiar when the UV theory is a weakly coupled Lagrangian field theory: the contributions to the anomaly coefficients simply come from the fermion degrees of freedom. Suppose the charge assignments

[28] The reader might be surprised that in reviewing c-extremization we have come upon c-maximization. The reason is that our argument is a little bit more refined than the one originally given in [47] and allows us to dismiss the possibility of extra right-moving currents in the IR. The argument in [47] has a loophole as far as right-moving currents are concerned: it is argued that the R-symmetry current must commute with all the currents in an N=2 SKM multiplet. While that is true for the top components of the SKM multiplet, it is not the case for the R-current of the N=2 SKM multiplet!

for the Fermi and chiral multiplets are as follows.

symmetry	Φ^a	Γ^A
$\mathrm{U}(1)_{\mathrm{R}}$	r_a	R_A
$\mathrm{U}(1)^\alpha$	q_a^α	Q_A^α

We then have [29]

$$a^{00} = \sum_A R_A^2 - \sum_a (r_a - 1)^2 , \qquad a^{0\beta} = \sum_A R_A Q_A^\beta - \sum_a (r_a - 1) q_a^\beta ,$$
$$M^{\alpha\beta} = \sum_A Q_A^\alpha Q_A^\beta - \sum_a q_a^\alpha q_a^\beta . \tag{2.8.30}$$

Another simple case where these coefficients are easily determined is when the UV theory is a relevant deformation of a compact and unitary (0,2) SCFT. In that case we can easily express the anomaly coefficients in terms of the normalizations of the UV currents. The R-current $\overline{J}$ is of course right-moving, and satisfies

$$-a^{00} = \frac{1}{3}\overline{c}^{\mathrm{uv}} = \overline{z}^2 \langle \overline{J}^{\mathrm{uv}}(\overline{z}) \overline{J}^{\mathrm{uv}}(0) \rangle . \tag{2.8.31}$$

The UV CFT may also have some left-moving currents U(1) currents J_α, $\alpha = 1, \ldots, r$, that satisfy for $z \neq 0$

$$M^{\alpha\beta} = z^2 \langle J_\alpha^{\mathrm{uv}}(z) J_\beta^{\mathrm{uv}}(0) \rangle , \qquad a^{0\beta} = \langle J_\alpha^{\mathrm{uv}}(z) \overline{J}^{\mathrm{uv}}(0) \rangle = 0 . \tag{2.8.32}$$

It is possible for the UV theory to also have a number of additional right-moving currents. These arise whenever the supercurrent multiplet $\overline{\mathcal{S}}$ defined in exercise (2.11) is decomposable:

$$\overline{\mathcal{S}} = \sum_{\dot{\alpha}=1}^{\overline{r}} \overline{\mathcal{S}}_{\dot{\alpha}} ,$$

where each $\overline{\mathcal{S}}_{\dot{\alpha}}$ generates a superconformal algebra with $\overline{c} = \overline{c}_{\dot{\alpha}}$, and the $\overline{\mathcal{S}}_{\dot{\alpha}} \overline{\mathcal{S}}_{\dot{\beta}}$ OPE is trivial for $\alpha \neq \beta$. This includes the case that the UV CFT has an N=2 SKM algebra — an extension of the N=1 SKM structure reviewed above. Suppose we now turn on a relevant deformation of this UV theory. As we saw, this means picking a set of chiral primary operators $\mathcal{U}$ with $h_{\mathcal{U}} = 1/2 + \overline{h}_{\mathcal{U}}$ and $\overline{h}_{\mathcal{U}} = \overline{q}_{\mathcal{U}}/2 < 1/2$.

A relevant deformation necessarily preserves every N=2 SKM algebra, simply because there are no N=2 SKM representations with $0 < \overline{q}_{\mathcal{U}} < 1$. To understand that point, note that an abelian N=2 SKM necessarily contains an anti-holomorphic field ψ of charge $\overline{q} = 1$, and any $\mathcal{U}$ must have an integer $\overline{q}_{\mathcal{U}}$ in order to have a local OPE with ψ. For a non-abelian

[29] The superspace coordinate θ has R-charge +1; this accounts for the $(r_a - 1)$ factors in the coefficients.

N=2 SKM the relevant deformation must involve a primary operator of the current algebra generated by the free fermions, and the dimensions of these satisfy the bound

$$\overline{h} \geq \frac{k}{2} \frac{\ell(\boldsymbol{r})}{\dim \boldsymbol{r}} \geq 1/2 \ . \tag{2.8.33}$$

While all the N=2 SKM components of the superconformal algebra must be preserved, it may be that the deformation also preserves a number of other $\overline{\mathcal{S}}_{\dot{\alpha}}$ components. Each such component will then lead to a factor in the IR superconformal algebra, and of course we do not need to work to find the R-symmetry charges for these factors: they are RG-invariant. So, without loss of generality we will now restrict the $\dot{\alpha}$ to run over the factors that do involve the deformation. More precisely, we can assume that for every $\dot{\alpha}$ there is some $\mathcal{U}$ charged with respect to $\overline{J}_{\dot{\alpha}}$. Once this is the case, the most general form for an R-current conserved along the RG flow is of the form

$$\mathcal{J}^{\text{uv}} = \sum_{\alpha=1}^{r} s^{\alpha} J_{\alpha}^{\text{uv}} \frac{\partial}{\partial z} + \overline{J}^{\text{uv}} \frac{\partial}{\partial \overline{z}} \ . \tag{2.8.34}$$

Here the s^{α} are to-be-determined coefficients, and $\overline{J}^{\text{uv}} = \sum_{\dot{\alpha}} \overline{J}_{\dot{\alpha}}$ is the diagonal R-current. This current will be conserved in the presence of the deformation if for every $\mathcal{U}$ we can find a solution to

$$\sum_{\alpha=1}^{r} q_{\mathcal{U}}^{\alpha} s_{\alpha} = 1 - \overline{q}_{\mathcal{U}} \ . \tag{2.8.35}$$

This is just the condition for the term $\Delta S = \int d^2 z d\theta \, \mathcal{U} + \text{h.c.}$ to be invariant under the $\mathcal{J}^{\text{uv}}$ symmetry.

Suppose that a solution exists, i.e. we can set $s_{\alpha} = \sigma_{\alpha}$ and preserve and R-symmetry current $\mathcal{J}^{\text{uv}}$ along the flow. The solution may of course be ambiguous: we can shift $\sigma \to \sigma + \omega$, where ω is in the kernel of the matrix of charges $q_{\mathcal{U}}^{\alpha}$, i.e. $\sum_{\alpha} q_{\mathcal{U}}^{\alpha} \omega_{\alpha} = 0$. Let us normalize the left-moving two-point functions so that $M^{\alpha\beta} = \delta^{\alpha\beta}$, and fix and orthonormal basis $\{\omega^1, \omega^2, \ldots, \omega^n\}$ for the kernel of $q_{\mathcal{U}}^{\alpha}$. Now the most general form of the coefficients s^{α} can be written as

$$s_{\alpha} = \sigma_{\alpha} + \sum_{i=1}^{n} t_i \omega_{\alpha}^{i} \tag{2.8.36}$$

for n real parameters t_i.

If the limit of the RG flow is a compact (0,2) CFT, then from the arguments given above we see that each ω_i will lead to a left-moving KM current. In addition, if there are no accidents along the flow, then $\mathcal{J}^{\text{uv}}$ must flow to the IR R-symmetry. The latter must have a zero two-point function with the former, and anomaly matching then uniquely determines the coefficients t_i:

$$t_i = -\sum_{\alpha=1}^{r} \omega_{\alpha}^{i} \sigma_{\alpha} \ , \tag{2.8.37}$$

so that the solution for the s_α is just given by a projection from σ to the component orthogonal to the ω_i: $s = \sigma_\perp = \sigma - \sum_i(\omega^i \cdot \sigma)\omega^i$. This fixes the IR R-charges and determines the central charge from the two-point function of the R-current:

$$\overline{c}^{\text{ir}} = \overline{c}^{\text{uv}} - 3\|\sigma_\perp\|^2 \ . \tag{2.8.38}$$

2.9 Torus partition functions and global anomalies

Chapter 3

Landau-Ginzburg theories

Abstract

In this chapter we study the simplest large class of (0,2) QFTs: the (0,2) Landau-Ginzburg theories. Our main aim is to introduce a number of key concepts relevant to general (0,2) theories in the context of these simple examples.

3.1 A class of Lagrangian theories

Recall that in chapter 1 we defined the (0,2) Yukawa theories. These are (0,2) Lagrangians where the interactions are encoded in terms of the holomorphic potentials E and J. For convenience, we review the construction of the Euclidean theory. We have n chiral superfields Φ^a, $a = 1, \ldots, n$, as well as N Fermi multiplets Γ^A, $A = 1, \ldots, N$, and the action is

$$S = \frac{1}{2\pi} \int d^2 z \ \left\{ \mathcal{D}\overline{\mathcal{D}} K_z + \frac{1}{\sqrt{2}} \mathcal{D}\, \mathcal{W} + \frac{1}{\sqrt{2}} \overline{\mathcal{D}}\, \overline{\mathcal{W}} \right\} . \tag{3.1.1}$$

$$K_z = \textstyle\sum_{a,\bar{b}} \frac{1}{4} \delta_{a,\bar{b}} \left[\overline{\Phi}^{\bar{b}} \partial \Phi^a - \Phi^a \partial \overline{\Phi}^{\bar{b}} \right] - \frac{1}{2} \sum_{A,\overline{B}} \delta_{A,\overline{B}} \overline{\Gamma}^{\overline{B}} \Gamma^A \tag{3.1.2}$$

is the free kinetic term,

$$\mathcal{W} = \textstyle\sum_A \Gamma^A J_A(\Phi) \tag{3.1.3}$$

is the holomorphic superpotential, and the Fermi multiplets obey the chirality condition

$$\overline{\mathcal{D}} \Gamma^A = E^A(\Phi) . \tag{3.1.4}$$

The theory is supersymmetric if and only if $\sum_A E^A J_A = 0$.

We say that a Yukawa theory is a Landau-Ginzburg theory (LG) theory if and only if it admits an R-symmetry $\mathfrak{u}(1)_{\mathrm{B}}$ with charges[1]

$$\begin{array}{cccc} & \theta & \Phi^a & \Gamma^A \\ \mathfrak{u}(1)_{\mathrm{B}} & +1 & 0 & +1 \end{array} \tag{3.1.5}$$

In such a theory $E^A = 0$, and the action is supersymmetric for any J; moreover, the theory automatically admits an $\mathcal{R}$-multiplet of supercurrents.

Beyond this, it appears that there is no completely agreed-upon terminology to distinguish different classes of models. Indeed, some authors would call any Yukawa theory a LG theory, and many would call any non-linear sigma model with superpotential couplings a Landau-Ginzburg theory. This seems a bit too broad to the author. Indeed, there are a number of additional criteria that usefully restrict the landscape of these Landau-Ginzburg theories to a large but perhaps more manageable set. We will now list these criteria along with some comments.

1. We can restrict attention to J_A that are polynomial in the fields. While non-polynomial J_A certainly play a role, especially in various effective descriptions derived from some more fundamental UV theory, a restriction to polynomials is quite natural and useful.

2. We will restrict attention to theories that preserve classical (0,2) SUSY, i.e. there is a point $\phi_* \in \mathbb{C}^n$ such that

   $$J_A(\phi_*) = 0 \qquad \text{for all } A\ .$$

   Our ultimate aim is to say something about the IR fixed point of the theory, and preserving SUSY along the RG flow is a fundamental simplification.

3. We will restrict attention to theories with a compact space of vacua. When this is not the case we must worry about the behavior of the path integral at large field values, and we might suspect that the generic situation will be a lack of normalizable ground state for the theory.

   When the J_A are polynomials the only possible compact holomorphic subset of $\mathbb{C}^n$ defined by the simultaneous vanishing of the J_A is a collections of points.

   We will say a LG theory is "compact" when the space of classical vacua is compact.

   Clearly we must then insist on $N \geq n$, since otherwise we simply have too few equations to fix the n expectation values of Φ^a.

4. Perhaps the most interesting restriction is to theories that admit another $\mathfrak{u}(1)$ symmetry, denoted $\mathfrak{u}(1)_{\mathrm{L}}$, with charges

   $$\begin{array}{cccc} & \theta & \Phi^a & \Gamma^A \\ \mathfrak{u}(1)_{\mathrm{L}} & 0 & q_a & Q_A \end{array} \tag{3.1.6}$$

[1] The reason for the name of this R-symmetry will become apparent soon—it reflects the fact that any such theory admits a B/2 twist.

This constrains the J_A to be quasi-homogeneous polynomials that satisfy

$$J_A(t^q\Phi) = t^{-Q_A} J_A(\Phi) \ . \tag{3.1.7}$$

Note that these charges are only defined up to an over-all scale. We will see how to fix the normalization below. As we will see shortly, this $\mathrm{U}(1)_{\mathrm{L}}$ symmetry will allow us to construct a UV candidate for the R-symmetry of the superconformal IR fixed point. Thus quasi-homogeneous and compact (0,2) LG theories offer a large landscape of theories where we may hope to use the simple UV description to capture properties of non-trivial SCFTs.

3.2 Currents and supercurrents

Every LG theory has an $\mathcal{R}$ supercurrent multiplet associated to the $\mathfrak{u}(1)_{\mathrm{B}}$ R-symmetry with components

$$\mathcal{R}_B = \sum_{A=1}^{N} \Gamma^A \overline{\Gamma}^A \ , \qquad \overline{\mathcal{R}}_B = -\frac{1}{2}\sum_{a=1}^{n} \overline{\mathcal{D}\Phi}^a \mathcal{D}\Phi^a \ ,$$
$$\mathcal{T}_B = -\sum_{a=1}^{n} \partial\Phi^a \partial\overline{\Phi}^a - \frac{1}{2}\sum_{A=1}^{N}\left(\overline{\Gamma}^A\partial\Gamma^A + \Gamma^A\partial\overline{\Gamma}^A\right) \ . \tag{3.2.1}$$

Recalling the equations of motion (see exercise 1.12) that follow from the Lagrangian with $E^A = 0$,

$$\mathcal{D}\Gamma^A = \sqrt{2}\overline{J}_A \ , \qquad \partial\mathcal{D}\Phi^a = \sqrt{2}\sum_{A=1}^{N} \overline{J}_{A,a}\overline{\Gamma}^A \ ,$$
$$\overline{\mathcal{D}\Gamma}^A = \sqrt{2}J_A \ , \qquad \partial\overline{\mathcal{D}\Phi}^a = \sqrt{2}\sum_{A=1}^{N} J_{A,a}\Gamma^A \ , \tag{3.2.2}$$

we can show that

$$\bar\partial\mathcal{R}_B + \partial\overline{\mathcal{R}}_B = 0 \ , \qquad \overline{\mathcal{D}}\left(\mathcal{T}_B - \tfrac{1}{2}\partial\mathcal{R}_B\right) = 0 \ . \tag{3.2.3}$$

When the superpotential is quasi-homogeneous, so that there is an $\mathfrak{u}(1)_{\mathrm{L}}$ global symmetry, there is also a current multiplet with components $\mathcal{K}$—a pure imaginary spin 1 operator, and Ψ—a chiral spin $-1/2$ field that satisfy (up to equations of motion)

$$\overline{\mathcal{D}}\mathcal{K} = -2\partial\Psi \ , \qquad \mathcal{D}\mathcal{K} = 2\partial\overline{\Psi} \ , \qquad \Longrightarrow \ \bar\partial\mathcal{K} + \partial(\mathcal{D}\Psi - \overline{\mathcal{D}\Psi}) = 0 \ . \tag{3.2.4}$$

A possible improvement term for the current multiplet is specified in terms of a pure imaginary superfield U:

$$\mathcal{K} \to \mathcal{K} - 2\partial U \ , \qquad \Psi \to \Psi + \overline{\mathcal{D}}U \ , \qquad \overline{\Psi} \to \overline{\Psi} - \mathcal{D}U \ . \tag{3.2.5}$$

Recall that in our Euclidean conventions reality is defined with respect to the action of the charge conjugation operator $\mathcal{C}$ defined in section 1.8 .

For the LG theory these take the form

$$\mathcal{K} = \sum_{A=1}^{N} Q_A \Gamma^A \overline{\Gamma}^A - \frac{1}{2}\sum_{a=1}^{n} q_a (\Phi^a \partial \overline{\Phi}^a - \overline{\Phi}^a \partial \Phi^a) \ ,$$
$$\Psi = -\frac{1}{4}\sum_{a=1}^{n} q_a \Phi^a \overline{\mathcal{D}\Phi}^a = \overline{\mathcal{D}}\left[-\frac{1}{4}\sum_{a=1}^{n} q_a \Phi^a \overline{\Phi}^a\right] \ , \qquad (3.2.6)$$

In particular, we see that $\Psi = \overline{\mathcal{D}}\mathcal{A}$ for a real superfield

$$\mathcal{A} = -\frac{1}{4}\sum_{a=1}^{n} q_a \Phi^a \overline{\Phi}^a \ . \qquad (3.2.7)$$

The existence of $(\mathcal{K}, \mathcal{A})$ means that the $\mathcal{R}$-multiplet is not unique: for any parameter t we can set

$$\mathcal{R}' = \mathcal{R}_B + t\mathcal{K} \ , \qquad \overline{\mathcal{R}}' = \overline{\mathcal{R}}_B + t[\mathcal{D}, \overline{\mathcal{D}}]\mathcal{A} \ , \qquad \mathcal{T}' = \mathcal{T}_B - t\partial^2 \mathcal{A} \qquad (3.2.8)$$

and obtain another $\mathcal{R}$-multiplet.

If there are multiple quasi-homogeneous conditions labeled by $\alpha = 1, \ldots, r$, we obtain a $\mathfrak{u}(1)^{\oplus r}$ symmetry with current multiplets $(\mathcal{K}^\alpha, \mathcal{A}^\alpha)$ and corresponding charges q^α_a and Q^α_A for the Φ^a and Γ^A respectively. In this case, there is a larger ambiguity in the $\mathcal{R}$-multiplet: now for any set of r parameters t_α we can set

$$\mathcal{R}' = \mathcal{R}_B + t_\alpha \mathcal{K}^\alpha \ , \qquad \overline{\mathcal{R}}' = \overline{\mathcal{R}}_B + t_\alpha [\mathcal{D}, \overline{\mathcal{D}}]\mathcal{A}^\alpha \ , \qquad \mathcal{T}' = \mathcal{T}_B - t_\alpha \partial^2 \mathcal{A}^\alpha \qquad (3.2.9)$$

and obtain another $\mathcal{R}$-multiplet.

All of this is offers a concrete example of our discussion of extremization in the previous chapter, and the resolution of the ambiguity is then much the same. If we assume that there are no accidents along the flow from the UV LG theory to a compact unitary SCFT, then c-extremization determines the coefficients t_α such that the $\mathcal{R}$ multiplet

$$\mathcal{R}^{\text{uv}} = \mathcal{R}_B + \sum_{\alpha=1}^{r} t_\alpha \mathcal{K}^\alpha \ , \quad \overline{\mathcal{R}}^{\text{uv}} = \overline{\mathcal{R}}_B + \sum_{\alpha=1}^{r} t_\alpha [\mathcal{D}, \overline{\mathcal{D}}]\mathcal{A}^\alpha \ , \quad \mathcal{T}^{\text{uv}} = \mathcal{T}_B - \sum_{\alpha=1}^{r} t_\alpha \partial^2 \mathcal{A}^\alpha \ , \quad (3.2.10)$$

flows in the IR to the generators of a left-moving Virasoro algebra and a right-moving N=2 super-Virasoro algebra. Moreover, as we argued quite generally above, in the accident-free scenario the remaining current multiplets must either decouple or flow to left-moving KM currents. Since the lowest components of the (super)current multiplets involved are the $\mathfrak{u}(1)^{\oplus r} \oplus \mathfrak{u}(1)_{\text{R}}$ conserved currents we can use the c-extremization technique to fix the coefficients t_α: we just need to restrict the coefficients in (2.8.30) to $R_A = 1$ and $r_a = 0$. The results are

$$a^{00} = N - n \ , \qquad a^{0\beta} = \sum_{A=1}^{N} Q^\beta_A + \sum_{a=1}^{n} q^\beta_a \ , \qquad M^{\alpha\beta} = \sum_{A=1}^{N} Q^\alpha_A Q^\beta_A - \sum_{a=1}^{n} q^\alpha_a q^\beta_a \ . \qquad (3.2.11)$$

Recall that in order for the accident-free scenario to hold M must have non-negative eigenvalues; moreover, all currents that lie in the kernel of M must decouple from the IR, so that we can restrict attention to currents in the orthogonal complement of $\ker M$. In order to keep the notation simple we will just assume that α runs over currents in this orthogonal complement, so that M is in fact a positive matrix. The parameters t_β are then fixed

$$t_\beta = -\sum_{\alpha=1}^{r} a^{0\alpha}(M^{-1})_{\alpha\beta} \ , \tag{3.2.12}$$

which also determines the IR central charge to be

$$\frac{\overline{c}^{\text{ir}}}{3} = n - N + \sum_{\alpha,\beta=1}^{r} a^{0\alpha}(M^{-1})_{\alpha\beta} a^{0\beta} \ . \tag{3.2.13}$$

Some simple conclusions follow from this extremization exercise. First, to have a putative accident-free IR SCFT we need $r > 0$: otherwise the central charge we obtain is non-sensical. If $r = 1$, then $\text{U}(1)_\text{L}$ is unique up to a rescaling, which is precisely the choice of parameter t. We can fix this to $t = 1$ by normalizing the charges so that

$$-\sum_{A=1}^{N} Q_A - \sum_{a=1}^{n} q_a = \sum_{A=1}^{N} Q_A^2 - \sum_{a=1}^{n} q_a^2 \ . \tag{3.2.14}$$

This condition was first described in (0,2) LG theories obtained from (0,2) GLSMs in [48].

For what follows it is helpful to think of extremization as determining a distinguished symmetry sub-algebra $\mathfrak{u}(1)_\text{L} \oplus \mathfrak{u}(1)_\text{R} \subset \mathfrak{u}(1)_\text{B} \oplus \mathfrak{u}^{\oplus r}$; $\mathfrak{u}(1)_\text{L}$ has the current multiplet

$$\mathcal{K} = \sum_\alpha t_\alpha \mathcal{K}^\alpha \ , \qquad\qquad \mathcal{A} = \sum_\alpha t_\alpha \mathcal{A}^\alpha \ , \tag{3.2.15}$$

and $\mathfrak{u}(1)_\text{R}$ corresponds to the R-current in the $\mathcal{R}^{\text{uv}}$ multiplet.

3.3 The chiral algebra of a LG theory

A remarkable feature of LG theories and (0,2) Lagrangian theories in general is the existence of the chiral algebra of holomorphic operators, essentially a holomorphic CFT. This structure was elucidated in the context of (2,2) LG theories in [49]; earlier explorations were made in [50]. It was applied to (0,2) LG and GLSM models in [48, 51–54]. The chiral algebra has since been explored at great length in the context of (0,2) non-linear sigma models: see, e.g. [55–58]. More recently, the structure has been revisited in the context of (0,2) LG theories in [59, 60].

The general idea

A theory with a (0,2) supersymmetry algebra, such as any theory with an $\mathcal{R}$-multiplet, has the supercharges $\boldsymbol{Q}$ and $\overline{\boldsymbol{Q}}$ that obey $\boldsymbol{Q}^2 = \overline{\boldsymbol{Q}}^2 = 0$ and $\{\boldsymbol{Q}, \overline{\boldsymbol{Q}}\} = -2\bar{\partial}$. The chiral algebra for the theory is obtained by restricting the operators to $\overline{\boldsymbol{Q}}$ cohomology:

$$H_{\overline{\boldsymbol{Q}}} = \frac{\ker \overline{\boldsymbol{Q}}}{\operatorname{im} \overline{\boldsymbol{Q}}} = \frac{\{\mathcal{O} \quad | \ \overline{\boldsymbol{Q}} \cdot \mathcal{O} = 0\}}{\{\mathcal{O} = \overline{\boldsymbol{Q}} \cdot \mathcal{X}\}} \ . \tag{3.3.1}$$

The OPE defines an algebra on $H_{\overline{\boldsymbol{Q}}}$. To see this, let $\mathcal{O}(z,\overline{z})$ be a local operator and $[\mathcal{O}(z,\overline{z})]$ be its cohomology class. We then see that[2]

$$-2\bar{\partial}\mathcal{O}(z,\overline{z}) = \overline{\boldsymbol{Q}} \cdot (\boldsymbol{Q} \cdot \mathcal{O}) + \boldsymbol{Q} \cdot (\overline{\boldsymbol{Q}} \cdot \mathcal{O}) = \overline{\boldsymbol{Q}} \cdot (\boldsymbol{Q} \cdot \mathcal{O}) \ . \tag{3.3.2}$$

Thus, the cohomology class $[\mathcal{O}(z,\overline{z})]$ is $\overline{z}$-independent, i.e. holomorphic! It follows that the OPE of $\overline{\boldsymbol{Q}}$–closed operators $\mathcal{O}_i$ takes the form

$$\mathcal{O}_1(z,\overline{z})\mathcal{O}_2(0) \sim \sum_i F_{12}{}^i(z)\mathcal{O}_i(0) + \overline{\boldsymbol{Q}}(\cdots) \tag{3.3.3}$$

and thus descends to an OPE in cohomology:

$$[\mathcal{O}_1](z)[\mathcal{O}_2](0) \sim \sum_i F_{12}{}^i(z)[\mathcal{O}_i](0) \ . \tag{3.3.4}$$

Lorentz invariance implies that the position dependence of the OPE is entirely determined by the spins of the operators:

$$[\mathcal{O}_1](z)[\mathcal{O}_2](0) \sim \sum_i C_{12}{}^i z^{s_i - s_1 - s_2}[\mathcal{O}_i](0) \ . \tag{3.3.5}$$

The (infinite–dimensional) subspace of operators $H_{\overline{\boldsymbol{Q}}}$, together with the holomorphic OPE, define the chiral algebra of our theory. In general the OPE coefficients $C_{12}{}^i$ depend on parameters of the theory, so that we actually get a family of chiral algebras.

A key property of the chiral algebra is scale–invariance. Although the original QFT need not be scale-invariant, the chiral algebra naturally is scale–invariant, simply because Lorentz invariance and holomorphy fix the OPE position dependence to be power–law. Thus, the chiral algebra is an RG–invariant of the QFT. This is very useful, since in the IR it encodes non-trivial information about an SCFT, but it may also be computed in the UV; when the UV description is weakly–coupled the structure is quite computable.

The reader should bear in mind that this invariance does not imply that the chiral algebra is determined by *classical* computations in the UV. It is quite possible to have a Lagrangian theory for which the classical chiral algebra determined by the Lagrangian differs from the quantum chiral algebra. An extreme example is offered by the $\mathbb{CP}^1$ sigma model,

[2] We use the short-hand $\overline{\boldsymbol{Q}} \cdot \mathcal{O}$ for $[\overline{\boldsymbol{Q}}, \mathcal{O}]$ or $\{\overline{\boldsymbol{Q}}, \mathcal{O}\}$, depending on whether $\mathcal{O}$ is bosonic or fermionic.

where the classical chiral algebra is infinite dimensional, while the quantum chiral algebra is trivial [55, 57, 61]. Moreover, accidental symmetries can also invalidate the relation between the UV and IR chiral algebras. However, we also have plenty of examples where classical or semi-classical UV computations do determine the chiral algebra of the theory, or at least some preferred finite-dimensional subspace.

It is often useful to present the chiral algebra in superspace. For that purpose it is convenient to note that $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are conjugate operators. That is,

$$\overline{\mathcal{Q}} = \exp[-2\theta\overline{\theta}\overline{\partial}]\overline{\mathcal{D}}\exp[2\theta\overline{\theta}\overline{\partial}] \ . \tag{3.3.6}$$

This means that $H_{\overline{\boldsymbol{Q}}}$ is isomorphic to $\overline{\mathcal{D}}$ cohomology. In particular, a field $\mathcal{O}$ is in $H_{\overline{\boldsymbol{Q}}}$ if and only if it is the lowest component of a chiral superfield; $\mathcal{O}$ is trivial in $H_{\overline{\boldsymbol{Q}}}$ if and only if it is the lowest component of a $\overline{\mathcal{D}}$-exact superfield.

Exercise 3.1. Show that the preceding statement is correct. First argue that every $\mathcal{O}$ is the lowest component of a unique superfield $\mathcal{S}_{\mathcal{O}}$; next, show that $\overline{\boldsymbol{Q}}\cdot\mathcal{O} = 0$ if and only if $\overline{\mathcal{D}}\mathcal{S}_{\mathcal{O}} = 0$, and $\mathcal{O} = \overline{\boldsymbol{Q}}\cdot X$ if and only if X is the lowest component of $\mathcal{S}_X$ with $\mathcal{S}_{\mathcal{O}} = \overline{\mathcal{D}}\mathcal{S}_X$. [3]

KMV algebra in cohomology

$H_{\overline{\boldsymbol{Q}}}$ may contain the stress tensor T. More precisely, suppose that the stress tensor components of the QFT satisfy

$$\overline{\boldsymbol{Q}}\cdot T = 0 \ , \qquad \overline{T} = \overline{\boldsymbol{Q}}\cdot\mathcal{X}_1 \ , \qquad \Theta = \overline{\boldsymbol{Q}}\cdot\mathcal{X}_2 \ . \tag{3.3.7}$$

In that case the cohomology class $[T]$ is holomorphic, and we obtain a holomorphic CFT structure on the chiral algebra. This structure may be extended by any additional KM symmetries or supersymmetries that belong to the cohomology. While this kind of structure is found in, for example, the classical non-linear sigma model, it certainly does not exist in a LG theory. Remarkably, however, in a LG theory there are operators in $H_{\overline{\boldsymbol{Q}}}$ that generate a KMV algebra. Namely, consider the superfields

$$\mathcal{T}_\chi = \mathcal{T}^{\text{uv}} - \tfrac{1}{2}\partial\mathcal{R}^{\text{uv}} \ , \qquad \mathcal{K}^\alpha_\chi = \mathcal{K}^\alpha + 2\partial\mathcal{A}^\alpha \ . \tag{3.3.8}$$

By construction these satisfy

$$\overline{\mathcal{D}}\mathcal{T}_\chi = 0 \ , \qquad \overline{\mathcal{D}}\mathcal{K}^\alpha_\chi = 0 \ . \tag{3.3.9}$$

[3] Further discussion may be found in [60].

Taking lowest components, we obtain

$$\begin{aligned} T_\chi &= T_0 - \tfrac{1}{2}\sum_{\alpha=1}^{r} t_\alpha \partial K_\chi^\alpha, \\ T_0 &= -\sum_{a=1}^{n} \partial\phi^a\partial\overline{\phi}^a - \sum_{A=1}^{N}\gamma^A\partial\overline{\gamma}^A\ , \\ K_\chi^\alpha &= \sum_{A=1}^{N} Q_A^\alpha \gamma^A\overline{\gamma}^A - \sum_{a=1}^{n} q_a^\alpha \phi^a\partial\overline{\phi}^a\ , \end{aligned} \tag{3.3.10}$$

where the t_α are fixed by c-extremization as in (3.2.12). These operators have the correct spins and dimensions to be, respectively, the left-moving energy-momentum tensor and left-moving spin 1 currents. However, they do not have the correct reality properties; for instance, K_χ^α is not pure imaginary!

Real or not, the OPE of these chiral operators can be computed in the free UV theory. This is simply because the superpotential W has an over-all dimensionful coupling μ, and, as we argued above, the OPE in $H_{\overline{\boldsymbol{Q}}}$ will be μ–independent.

Exercise 3.2. Use the free field OPEs

$$\gamma^A(z_1)\overline{\gamma}^B(z_2) \sim \frac{\delta^{AB}}{z_{12}}\ , \qquad \phi^a(z_1)\partial\overline{\phi}^b(z_2) \sim \frac{\delta^{ab}}{z_{12}} \tag{3.3.11}$$

and (3.2.12) to evaluate the OPEs of T_χ and K_χ^α. The result is

$$\begin{aligned} K_\chi^\alpha(z_1)K_\chi^\beta(z_2) &\sim \frac{M^{\alpha\beta}}{z_{12}^2}\ , \\ T_\chi(z_1)K_\chi^\alpha(z_2) &\sim \frac{K_\chi^\alpha(z_2)}{z_{12}} + \frac{\partial K_\chi^\alpha(z_2)}{z_{12}^2}\ , \\ T_\chi(z_1)T_\chi(z_2) &\sim \frac{c/2}{z_{12}^4} + \frac{2T_\chi(z_2)}{z_{12}^2} + \frac{\partial T_\chi(z_2)}{z_{12}}\ , \end{aligned} \tag{3.3.12}$$

where

$$c = N - n + \overline{c}\ , \qquad \overline{c} = 3(n-N) + 3(a^0)^T M^{-1} a^0\ , \tag{3.3.13}$$

and $M^{\alpha\beta}$ and $a^{0\beta}$ are the anomaly coefficients in (3.2.11). This is precisely what we expect based on c-extremization: we find the central charge obtained above, as well as a left-moving $\mathfrak{u}(1)^{\oplus r}$ KM symmetry.

It is therefore tempting to assume that this structure really computes $H_{\overline{\boldsymbol{Q}}}$ of the IR SCFT. Barring accidents, this is a sound assumption, and it allows us to determine the charges and scaling dimensions for all operators in the IR $H_{\overline{\boldsymbol{Q}}}$ through computations in the weakly coupled (indeed, free) UV theory.

Chiral CFT beyond LG

The existence of the stress tensor in the chiral algebra of a LG theory depended very weakly on the details of the Lagrangian. Indeed, in any UV theory with an $\mathcal{R}$–supercurrent multiplet there is a candidate operator for the energy momentum tensor in $H_{\overline{\boldsymbol{Q}}}$:

$$\mathcal{T}_\chi = \mathcal{T}^{\text{uv}} - \tfrac{1}{2}\partial\mathcal{R}^{\text{uv}} \ . \qquad (3.3.14)$$

Similarly, whenever the UV theory has current multiplets $(\mathcal{K},\mathcal{A})$ that have the structure as above, we expect to find candidate operators for KM currents in $H_{\overline{\boldsymbol{Q}}}$. Barring accidents, these should describe the holomoprhic sector of the IR SCFT. Note that the $\mathbb{CP}^1$ model mentioned above is not a counter-example: that theory has no $\mathcal{R}$–multiplet.

It may also be possible to find a UV representation for a more elaborate holomorphic structure. For instance, in the case of (2,2) LG theories, there is an N=2 superconformal algebra in $\overline{\boldsymbol{Q}}$ cohomology [49], and this can also be generalized to (2,2) gauge theories.

3.4 Topological heterotic rings : general structure

In this section we identify a class of operators in (0,2) theories introduced in [56, 62] that closely mimic the cc and ac rings of (2,2) theories. The notions coincide in (2,2) theories, and when a (2,2) theory is smoothly deformed to a more general (0,2) theory then under some mild assumptions the cc and ac rings are deformed to topological heterotic rings. The study of these rings has provided many insights into the structure of (0,2) theories.

Consider a (2,2) SCFT. In this case, as we explained earlier, the CFT Hilbert space contains subspaces H_{cc} and H_{ac} (as well as their complex conjugates) that are finite dimensional rings graded by $\mathfrak{u}(1)_{\text{L}} \oplus \mathfrak{u}(1)_{\text{R}}$. Using the supercharge $\overline{\boldsymbol{Q}} = \overline{G}^{+}_{-1/2}$ it is easy to see that these rings are isomorphic to finite-dimensional sub-spaces of $H_{\overline{\boldsymbol{Q}}}$ defined as follows:

$$\begin{aligned} H_{\text{cc}} &\simeq H_{B/2} = \{\mathcal{O} \in H_{\overline{\boldsymbol{Q}}} \ \mid \ h_{\mathcal{O}} - q_{\mathcal{O}}/2 = 0\} \ , \\ H_{\text{ac}} &\simeq H_{A/2} = \{\mathcal{O} \in H_{\overline{\boldsymbol{Q}}} \ \mid \ h_{\mathcal{O}} + q_{\mathcal{O}}/2 = 0\} \ . \end{aligned} \qquad (3.4.1)$$

We obtain the ring structure as in (2.4.30). [4] For instance, if $\mathcal{O}_i \in H_{B/2}$, then

$$\mathcal{O}_i(z_1)\mathcal{O}_j(0) \sim \sum_s C_{ij}{}^{s} z^{h_s - q_s/2}\mathcal{O}_s(0) \ , \qquad (3.4.2)$$

where $\mathcal{O}_s \in H_{\overline{\boldsymbol{Q}}}$. Since the (2,2) theory has a unitarity bound $h \geq q/2$ for all operators, it follows that the OPE is non-singular, and taking $z \to 0$ we find that the only operators that contribute are $\mathcal{O}_k \in H_{B/2}$. The result is the ring structure $\mathcal{O}_i\mathcal{O}_j = \sum_k C_{ij}{}^{k}\mathcal{O}_k$.

So far, we just reworded the familiar structure of the cc ring of a (2,2) theory. We will now show that in a large class of theories the ring structure on $H_{B/2}$ will continue to exist for general marginal deformations. That will include any (0,2) marginal deformations of (2,2) theories but will also apply to any (0,2) theory with the following properties:

[4] In what follows we will omit the distinction between $\overline{\boldsymbol{Q}}$-closed operator $\mathcal{O}$ and its cohomology class $[\mathcal{O}] \in H_{\overline{\boldsymbol{Q}}}$, and we will drop $\overline{\boldsymbol{Q}}$–exact terms in the OPEs.

1. the spin is integer or half–integer: $h - \overline{h} \in \frac{1}{2}\mathbb{Z}$;
2. $q - \overline{q} \in \mathbb{Z}$, and there is a fermion number $(-1)^F = (-1)^{q-\overline{q}}$;
3. spin-statistics holds: the spin is half-integer for fermions and integer for bosons;
4. there is a point in the moduli space where the spectrum of local operators satisfies the bound $h \geq \frac{1}{2}q$.

From these assumptions it follows that for any operator we have

$$h - \overline{h} - \tfrac{1}{2}(q - \overline{q}) = m \in \mathbb{Z} \;, \tag{3.4.3}$$

and for any operator in $H_{\overline{\boldsymbol{Q}}}$ the quantity $h - \frac{1}{2}q$ is a non-negative integer.

Consider a marginal deformation that preserves (0,2) superconformal invariance and the $\mathfrak{u}(1)_{\rm L}$ symmetry. While the dimensions and charges may shift, the integer m will remain unchanged for all operators. Thus, the bound $h \geq \frac{1}{2}q$ of the undeformed theory will continue to hold in $\overline{\boldsymbol{Q}}$–cohomology under any (0,2) marginal deformation that preserves $\mathfrak{u}(1)_{\rm L}$. It follows that the ring structure on $H_{B/2}$ will persist for any marginal $\mathfrak{u}(1)_{\rm L}$–preserving deformation off the (2,2) locus.

The SCFT may have more structure, where not only is $q - \overline{q} \in \mathbb{Z}$, but also $q \in \mathbb{Z}$. In that case we also have

$$h - \overline{h} + \tfrac{1}{2}(q + \overline{q}) = m' \in \mathbb{Z} \;, \tag{3.4.4}$$

and an exactly analogous argument will show that $H_{A/2}$ will persist for any marginal $\mathfrak{u}(1)_{\rm L}$–preserving deformation.

More generally, if we have a family of (0,2) SCFTs with $\mathfrak{u}(1)_{\rm L}$ as above, and a point p in the moduli space where $m \geq 0$ for all $\mathcal{O} \in H_{\overline{\boldsymbol{Q}}}$, then our theory has the $H_{B/2}$ ring. If, in addition $q \in \mathbb{Z}$ for this family, we also have the $H_{A/2}$ ring. The existence of p sounds like a strong assumption, but we can see that there is a class of (0,2) SCFTs where unitarity implies the desired bound. Suppose the $\mathfrak{u}(1)_{\rm L}$ current has level r. We then have the Sugawara unitarity bound

$$h \geq \frac{q^2}{2r} \;, \tag{3.4.5}$$

and it follows that

$$h - \frac{q}{2} \geq \frac{q(q-r)}{2r} \geq -\frac{r}{8} \;. \tag{3.4.6}$$

Hence, for $r < 8$ we automatically have the desired bound: $m \geq 0$ for all $\mathcal{O} \in H_{\overline{\boldsymbol{Q}}}$. The same bound holds for m' whenever $q \in \mathbb{Z}$.

The dimensions of $H_{A/2}$ and $H_{B/2}$ may vary over the SCFT moduli space when states leave or descend to $H_{\overline{\boldsymbol{Q}}}$ cohomology. Suppose we consider a non-singular point p in the moduli space. As we move away a small distance ϵ, it may well be that a pair of states in

$H_{\overline{Q}}$ pair up and leave the cohomology for any $\epsilon > 0$. However, a finite deformation is required for a non-chiral state at p to decompose into two chiral states in cohomology. Therefore, any non-singular point in the moduli space has an open neighborhood U_p such that for all points $s \in U_p$ $\dim H_{\mathrm{B}/2}(s) \leq \dim H_{\mathrm{B}/2}(p)$ (and similarly for the A/2 ring). For a generic p the statement will hold with equality.

The statements just made apply to any massive (0,2) theory with an $\mathcal{R}$-multiplet. Any such theory will have a chiral algebra structure on $H_{\overline{Q}}$ with stress tensor as in (3.3.14). The topological heterotic ring(s) provide an easily accessible class of RG-invariants of such (0,2) theories. The simplest example of this structure is afforded by the LG theories, and we will now examine it in detail. Along the way, we will also start to develop some algebraic and geometric notion that will prove useful in the sequel.

3.5 Topological heterotic ring of (0,2) LG theory

As we saw in chapter 1, after integrating out the auxiliary fields, we obtain the following Euclidean action for (0,2) LG theory %%stupid up down a index!:

$$S = \frac{1}{2\pi} \int d^2 z \left\{ \bar{\partial} \overline{\phi}^a \partial \phi^a + \overline{\psi}^a \partial \psi^a + \overline{\gamma}_A \bar{\partial} \gamma^A - \gamma^A J_{A,a} \psi^a - \overline{\psi}^a \overline{J}^A_{,a} \overline{\gamma}_A + J_A \overline{J}^A \right\} . \qquad (3.5.1)$$

We lowered the index $\overline{\gamma}^A \to \overline{\gamma}_A$ for future convenience. Note that we are using the summation convention to simplify the notation; we will do this throughout this section. This action is of course supersymmetric and in particular invariant under the action of $\overline{Q}$; $\overline{Q}$ leaves $\phi^a, \overline{\psi}^a$ and γ^A invariant, while acting on the remaining fields as

$$\overline{Q} \cdot \overline{\phi}^a = \overline{\psi}^a \ , \qquad \overline{Q} \cdot \psi^a = -\bar{\partial} \phi^a \ , \qquad \overline{Q} \cdot \overline{\gamma}_A = -J_A \qquad (3.5.2)$$

For any J_A the theory has an $\mathcal{R}_B$-multiplet with a conserved current corresponding to the symmetry with charges

$$\begin{array}{cccc} & \phi^a & \psi^a & \gamma^A \\ \mathfrak{u}(1)_{\mathrm{B}} & 0 & -1 & +1 \end{array} \qquad (3.5.3)$$

We are interested in the $\overline{Q}$ cohomology $H_{\overline{Q}}$, or more specifically the topological heterotic sub–ring $H_{B/2} \subset H_{\overline{Q}}$. As we already explained quite generally, every cohomology class will have a representative without any $\bar{\partial}$ derivatives; a look at the action of $\overline{Q}$ then also shows that every cohomology class will have a representative without any ψ^a insertions. As we will see shortly, to describe $H_{B/2}$ we can also restrict to operators without any holomorphic derivatives, i.e. every class in $H_{B/2}$ has a representative in the space of operators $\mathcal{H}$ given by linear combinations of

$$\mathcal{O}[\omega^{k,l,s}] = \omega^{A_1 \cdots A_k}_{B_1 \cdots B_l; a_1 \cdots a_s}(\phi, \overline{\phi}) \overline{\gamma}_{A_1} \cdots \overline{\gamma}_{A_k} \overline{\psi}^{a_1} \cdots \overline{\psi}^{a_s} \gamma^{B_1} \cdots \gamma^{B_l} \ . \qquad (3.5.4)$$

The action of $\overline{\boldsymbol{Q}}$ on $\mathcal{H}$ is given by the differential operator

$$\overline{\boldsymbol{Q}} = \left[\sum_{a=1}^{n} \overline{\psi}^a \frac{\partial}{\partial \overline{\phi}^a} - \sum_{A=1}^{N} J_A \frac{\partial}{\partial \overline{\gamma}_A} \right] . \tag{3.5.5}$$

Note that $\overline{\boldsymbol{Q}}$ preserves the number of γ^A insertions, so that in evaluating the cohomology we can work at fixed l. By construction $\overline{\boldsymbol{Q}}$ has $\mathfrak{u}(1)_{\rm B}$ charge $+1$, and we can grade the $\mathcal{H}$ by l and the $\mathfrak{u}(1)_{\rm B}$ charge $\overline{q}_0$:

$$\mathcal{H} = \bigoplus_{l,\overline{q}_0} \mathcal{H}^{l,\overline{q}_0} . \tag{3.5.6}$$

So, at every l we are then interested in the cohomology of the complex

$$\cdots \xrightarrow{\quad \overline{\boldsymbol{Q}} \quad} \mathcal{H}^{l,\overline{q}_0-1} \xrightarrow{\quad \overline{\boldsymbol{Q}} \quad} \mathcal{H}^{l,\overline{q}_0} \xrightarrow{\quad \overline{\boldsymbol{Q}} \quad} \mathcal{H}^{l,\overline{q}_0+1} \xrightarrow{\quad \overline{\boldsymbol{Q}} \quad} \cdots \tag{3.5.7}$$

We will now show that every cohomology class in $H_{\overline{\boldsymbol{Q}}} \cap \mathcal{H}$ has a representative without any $\overline{\psi}$ insertions. It is not too hard to argue for it directly, but we will do it by using a more general set-up of the problem that employs bundle-valued differential forms.[5]

A geometric set-up for the cohomology

Let $E \to M$ be a holomorphic bundle of rank N over a complex manifold M of dimension $\dim_{\mathbb{C}} M = n$, and suppose E has a global section J. Let

$$\Omega^{k,s} = \Gamma(\wedge^k E^\vee \otimes \wedge^s (T_M^{(0,1)})^\vee), \tag{3.5.8}$$

That is, $\Omega^{k,s}$ is the space of $(0,s)$ differential forms valued in the k-th asymmetric power of the dual bundle $E^\vee$. Clearly $\Omega^{k,s}$ is non-empty only if $0 \leq k \leq N$ and $0 \leq s \leq n$, but it is often convenient to consider all integer k, s, with understanding that $\Omega^{k,s}$ is zero outside of that range.

If we fix a local holomorphic frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\}$ for $E^\vee$ and coordinates $(z^i, \overline{z}^{\overline{\imath}})$ for M, then $\omega \in \Omega^{k,s}$ takes the form

$$\omega = \frac{1}{k!s!} \boldsymbol{e}_{A_1} \wedge \cdots \wedge \boldsymbol{e}_{A_k} \omega^{A_1 \cdots A_k}_{\overline{\imath}_1 \cdots \overline{\imath}_s} d\overline{z}^{\overline{\imath}_1} \wedge \cdots \wedge d\overline{z}^{\overline{\imath}_s} . \tag{3.5.9}$$

We then have the standard definition for the Dolbeault operator: $\bar{\partial} : \Omega^{k,s} \to \Omega^{k,s+1}$:

$$\bar{\partial}\omega = \frac{1}{k!s!} \boldsymbol{e}_{A_1} \wedge \cdots \wedge \boldsymbol{e}_{A_k} \frac{\partial \omega^{A_1 \cdots A_k}_{\overline{\imath}_1 \cdots \overline{\imath}_s}}{\partial \overline{z}^{\overline{\imath}_0}} d\overline{z}^{\overline{\imath}_0} \wedge d\overline{z}^{\overline{\imath}_1} \wedge \cdots \wedge d\overline{z}^{\overline{\imath}_s} . \tag{3.5.10}$$

[5] We will explore these structures in greater detail in the next chapter; here we will just need some of the most basic algebraic concepts and definitions. The exercises provide some practice with these tools.

We also define the contraction $J\llcorner\omega \in \Omega^{k-1,s}$ via

$$J\llcorner\omega = \frac{1}{(k-1)!s!}\boldsymbol{e}_{A_2} \wedge \cdots \wedge \boldsymbol{e}_{A_k} J_{A_1} \omega^{A_1 A_2 \cdots A_k}_{\bar{\imath}_1 \cdots \bar{\imath}_s} d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{\bar{\imath}_s} \ . \tag{3.5.11}$$

By definition the contraction $J\llcorner$ gives zero on $\Omega^{0,s}$. With these definitions we construct the operator

$$\begin{aligned} d_J &: \Omega^{k,s} \to \Omega^{k,s+1} \oplus \Omega^{k-1,s} \\ d_J &: \omega \mapsto (-1)^k \bar{\partial}\omega - J\llcorner\omega \ , \end{aligned} \tag{3.5.12}$$

and we observe that $(d_J)^2 = 0$.

Exercise 3.3. Verify that with these definitions $(d_J)^2 = 0$. Also show that if we define for any integer r

$$\Omega^{(r)} = \bigoplus_{\substack{k,s \\ k-s=r}} \Omega^{k,s} \tag{3.5.13}$$

then d_J is a linear operator $d_J : \Omega^{(r)} \to \Omega^{(r-1)}$. Finally, check that if we identify the $\omega \in \Omega^{k,s}$ with operators of the form given in (3.5.4), then the action of d_J is the same as that of $\overline{\boldsymbol{Q}}$ in (3.5.5), and the grading by $\overline{q}_0$ is equivalent to the grading by $-r$.

The exercise shows that the cohomology of the complex in (3.5.7) is equivalent to the cohomology of the operator d_J on the chain complex built from the $\Omega^{(r)}$ [6]

$$H_{\overline{\boldsymbol{Q}}} \cap \mathcal{H}^{\overline{q}_0} = H^{(-\overline{q}_0)}_{d_J} = \frac{\ker\{d_J : \Omega^{(-\overline{q}_0)} \to \Omega^{(-\overline{q}_0-1)}\}}{\operatorname{im}\{d_J : \Omega^{(1-\overline{q}_0)} \to \Omega^{(-\overline{q}_0)}\}} \ . \tag{3.5.14}$$

The result is quite general and applies to a wide class of (0,2) theories with a superpotential defined on a non-trivial target space. We shall have more to say of these in the next chapters. For now, we will just consider the vasty simpler case of LG theories, where $M = \mathbb{C}^n$ and $E = \mathbb{C}^N \times \mathbb{C}^n$. Now the d_J cohomology reduces to an algebraic problem because Dolbeault cohomology of $\mathbb{C}^n$ is very simple: any $\bar{\partial}$-closed form of degree $s > 0$ is exact; a degree $s = 0$ form, i.e. a function, is $\bar{\partial}$–closed if and only if it is holomorphic. Consider now some d_J-closed polyform $\omega^{(r)} \in \Omega^{(r)}$

$$\omega^{(r)} = \sum_{\substack{k,s \\ k-s=r}} \omega^{k,s} = \cdots + \omega_{\text{top}} \ , \tag{3.5.15}$$

where $\omega_{\text{top}} \neq 0$ is the component of the polyform $\omega^{(r)}$ with largest value of s, s_{top}. A look at figure 3.1 shows that $d_J\omega^{(r)} = 0$ requires $\bar{\partial}\omega_{\text{top}} = 0$, and if $s_{\text{top}} > 0$, then this in turn means that there exists an η such that $\omega_{\text{top}} = \bar{\partial}\eta$. Now let

[6] We suppress the l label, since it does not play any role in the cohomology computation.

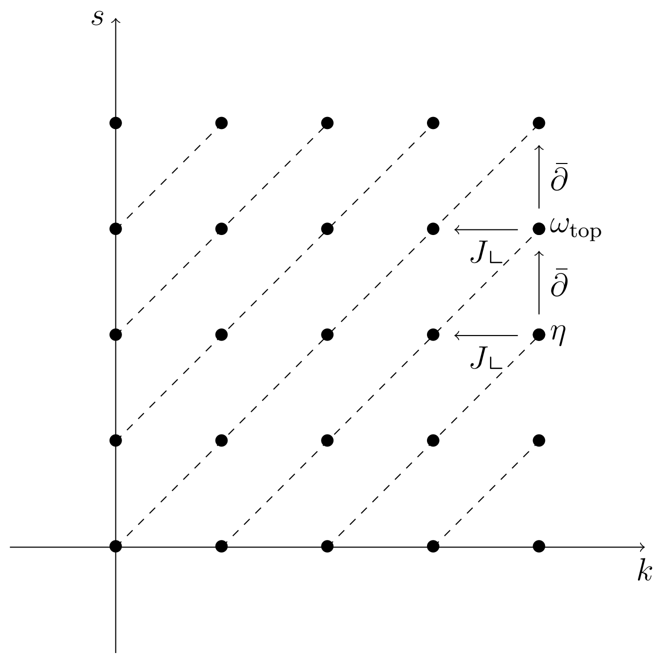

Figure 3.1: The spectral sequence for topological heterotic ring of (0,2) LG theory. The solid points label the $\Omega^{k,s}$, and the dashed lines the $\Omega^{(r)}$.

$$\omega'^{(r)} = \omega^{(r)} - d_J(-1)^{k_{\text{top}}}\eta \ . \tag{3.5.16}$$

Since they differ by an exat term, $\omega'^{(r)}$ and $\omega^{(r)}$ belong to the same d_J–cohomology class, but by construction ω' has its largest value of s at most $s_{\text{top}}-1$. Repeating this construction until the top form has $s=0$, we therefore show that

$$H^{(r)}_{d_J} = \frac{\ker\{J\llcorner\colon \widetilde{\Omega}^r \to \widetilde{\Omega}^{r-1}\}}{\text{im}\{J\llcorner\colon \widetilde{\Omega}^{r+1} \to \widetilde{\Omega}^r\}} \ , \tag{3.5.17}$$

where

$$\widetilde{\Omega}^r = \text{span}\{\omega^{A_1\cdots A_r}(\phi)e_{A_1}\wedge e_{A_2}\wedge\cdots e_{A_k}\} \ . \tag{3.5.18}$$

The cohomology $H^{(r)}_{d_J}$ is familiar in algebraic geometry [63–65]. To describe it, we introduce a few more objects. Let $R = \mathbb{C}[\phi_1,\phi_2,\ldots,\phi_n]$ be the ring of polynomials in n variables $\phi_1,\ldots,\phi_n$ with complex coefficients, and take $\mathcal{E} = R^{\oplus N}$ to be the rank N free module over R. It follows that $\widetilde{\Omega}^r = \wedge^r\mathcal{E}$. Finally, let $\boldsymbol{J} = \langle J_1, J_2,\ldots,J_N\rangle \subset R$ be an ideal in R with generators $J_1, J_2,\ldots,J_N$. We now define the Koszul complex for the ideal $\boldsymbol{J}\subset R$ as

$$K_\bullet = \ \cdots \xrightarrow{\ J\llcorner\ } \widetilde{\Omega}^{r+1} \xrightarrow{\ J\llcorner\ } \widetilde{\Omega}^{r} \xrightarrow{\ J\llcorner\ } \widetilde{\Omega}^{r-1} \xrightarrow{\ J\llcorner\ } \cdots \ . \tag{3.5.19}$$

In this terminology the cohomology groups we are after are identified with the homology groups of this complex—$H_k(K_\bullet, J\llcorner)$. This is the Koszul homology corresponding to ideal $\boldsymbol{J} \subset R$.[7]

Since $\wedge^0\mathcal{E} = R$, it follows that the complex terminates on the right with

$$\cdots \xrightarrow{\ J\llcorner\ } \wedge^2\mathcal{E} \xrightarrow{\ J\llcorner\ } \mathcal{E} \xrightarrow{\ J\llcorner\ } R \xrightarrow{\ J\llcorner\ } 0 \quad , \tag{3.5.20}$$

and therefore

$$H_0(K_\bullet, J\llcorner) = R/\boldsymbol{J} \ . \tag{3.5.21}$$

In other words, $H_0(K_\bullet, J\llcorner)$ is the quotient ring associated to the ideal.

A basic algebraic feature of these homology groups is that they measure the dimension of the ideal $\boldsymbol{J}$. It is easy to get an intuitive notion of the dimension of an ideal: the vanishing locus J_A in $\mathbb{C}^n$ may be a smooth sub-manifold, and in that case the dimension of the ideal is simply the dimension of the sub-manifold. More generally the vanishing locus is an algebraic variety and typically is singular as a manifold; however, there are algebraic notions of dimension that generalize to that more general setting [63, 64]. One of these is provided by the Koszul complex:

$$H_k(K_\bullet, J\llcorner) = \begin{cases} 0 & \text{for} \quad k > N - n + \dim(\boldsymbol{J}) \\ \text{non-zero} & \text{for} \quad k \leq N - n + \dim(\boldsymbol{J}) \ . \end{cases} \tag{3.5.22}$$

Additional features are illustrated in the following two exercises.

Exercise 3.4. Show that the $\wedge$ operation gives the space $\bigoplus_k H_k(K_\bullet, J\llcorner)$ a ring structure. That is, suppose we have classes $[\omega_a] \in H_{k_a}(K_\bullet, J\llcorner)$ for $a = 1, 2$, with representatives $\omega_a \in \widetilde{\Omega}^{k_a}$. Show that $\omega_1 \wedge \omega_2$ is annihilated by $J\llcorner$, and that $[\omega_1 \wedge \omega_2] \in H_{k_1+k_2}(K_\bullet, J\llcorner)$ is independent of representatives, i.e. shifting $\omega_a \to \omega_a + J\llcorner\eta_a$ for any $\eta_a \in \widetilde{\Omega}^{k_a-1}$ does not change $[\omega_1 \wedge \omega_2]$.

Exercise 3.5. Let $\epsilon_{A_1 \ldots A_N}$ denote the fully anti-symmetric rank N tensor with $\epsilon_{123\cdots N} = 1$. Show that this gives an isomorphism $\wedge^k\mathcal{E} = \wedge^{N-k}\mathcal{E}^\vee$, where $\mathcal{E}^\vee$ is the dual module to $\mathcal{E}$, where the explicit map is the "Hodge dual"

$$\begin{aligned} * : \wedge^k\mathcal{E} \quad &\to \wedge^{N-k}\mathcal{E}^\vee \ , \\ * : \omega^{B_1\cdots B_k} &\mapsto \tfrac{1}{k!}\omega^{B_1\cdots B_k}\epsilon_{B_1\cdots B_k A_1\cdots A_{N-k}} \ . \end{aligned}$$

Note that $\wedge^k\mathcal{E}$ and $\wedge^k\mathcal{E}^\vee$ are dual in the sense that each element in $\widetilde{\omega} \in \wedge^k\mathcal{E}^\vee$ yields an R–linear map from $\wedge^k\mathcal{E}$ to R, where $\omega \in \wedge^k\mathcal{E}$ is sent to $\widetilde{\omega}\llcorner\omega \in R$. A more mathematical notation for this statement is $\mathrm{Hom}(\wedge^k\mathcal{E}, R) = \wedge^k\mathcal{E}^\vee$, and our isomorphism allows us to identify this group with $\wedge^{N-k}\mathcal{E}$.

[7] This is a simple (perhaps simplest?) example of a more general structure in algebraic geometry, where R is replaced with some more general sheaf over a complex variety.

Now consider the dual Koszul complex

$$K^\bullet = \ \cdots \xrightarrow{J\wedge} \wedge^{k-1}\mathcal{E}^\vee \xrightarrow{J\wedge} \wedge^k\mathcal{E}^\vee \xrightarrow{J\wedge} \wedge^{k+1}\mathcal{E}^\vee \xrightarrow{J\wedge} \cdots \quad , \tag{3.5.23}$$

where the map $J\wedge$ is

$$\begin{aligned} J\wedge : \wedge^k\mathcal{E}^\vee \quad &\to \wedge^{k+1}\mathcal{E}^\vee \ , \\ J\wedge : \widetilde{\omega}_{B_1\cdots B_k} &\mapsto (k+1)J_{[B_1}\widetilde{\omega}_{B_2\cdots B_{k+1}]} \ , \end{aligned}$$

and the $[\cdots]$ brackets denote total anti-symmetrization, i.e. $T_{[AB]} = \frac{1}{2}(T_{AB} - T_{BA})$ and so forth. Show that $*$ induces an isomorphism of the cohomology groups of (3.5.23), $H^k(K^\bullet, J\wedge)$ to the previously defined $H_k(K_\bullet, J\llcorner)$, with isomorphism given by $*$ given above. To do so, prove the useful identities that $*^2 \wedge^k \mathcal{E} = (-1)^{k(N-k)} \wedge^k \mathcal{E}$ and that for any $\omega \in \wedge^k\mathcal{E}$ we have the identity

$$*(J\llcorner\omega) = (-1)^{k-1}(J \wedge *\omega) \ . \tag{3.5.24}$$

Finally, show that the induced map on the cohomology classes is independent of representatives.

The LG topological heterotic ring and Koszul cohomology

Having built up some algebraic machinery, we now return to the LG theory. As shown in the preceding section each of the $\overline{\boldsymbol{Q}}$ cohomology groups $\mathcal{H}^{l,\overline{q}_0}$ is isomorphic to the Koszul homology group $H_{-\overline{q}_0}(K_\bullet, J\llcorner)$. We will now argue that $\mathcal{H}^{0,\overline{q}_0}$ is the topological heterotic ring of LG theory.

Our discussion so far of the $\overline{\boldsymbol{Q}}$ cohomology has been carried out entirely in the classical theory. To work in the quantum theory we need to make sense of the composite operators that appear in (3.5.4). Fortunately, for the purpose of computing the $\overline{\boldsymbol{Q}}$ cohomology, we can use the free UV theory with chiral OPEs of (3.3.11) to define local operators that represent each cohomology class:

$$\mathcal{O}[\omega^{k,l}](z) =: \omega^{A_1\cdots A_k}_{B_1\cdots B_l}(\phi)\overline{\gamma}_{A_1}\cdots\overline{\gamma}_{A_k}\gamma^{B_1}\cdots\gamma^{B_l} : (z) \ . \tag{3.5.25}$$

Note that since $\phi(z_1)\phi(z_2)$ is non-singular as $z_1 \to z_2$, there are no ordering issues in constructing $\omega^{A_1\cdots A_k}_{B_1\cdots B_l}(\phi)$. Similarly, because the $\overline{\gamma}_A(z_1)\overline{\gamma}_B(z_2)$ is non-singular as $z_1 \to z_2$, normal ordering is only necessary when $l > 0$. The action of $\overline{\boldsymbol{Q}}$ on these operators is represented by

$$\overline{\boldsymbol{Q}} = \oint \frac{dz}{2\pi i} J_A(\phi)\gamma^A \ . \tag{3.5.26}$$

The symmetry group of our LG theory contains $\mathfrak{u}(1)_{\mathrm{B}} \oplus \mathfrak{u}(1)^{\oplus r}$ as in section 3.2, with charge assignments as follows.

(3.5.27)

	θ	Φ^a	Γ^A
$\mathfrak{u}(1)_{\mathrm{B}}$	1	0	1
$\mathfrak{u}(1)^\alpha$	0	q^α_a	Q^α_A

We grade the space of operators by these charges and denote the $\mathfrak{u}(1)_{\rm R}$ and $\mathfrak{u}(1)^\alpha$ charges of an operator $\mathcal{O}$ by, respectively, $\overline{q}_B[\mathcal{O}]$ and $q^\alpha[\mathcal{O}]$. Once we assume the RG flow is accident free, we also find a distinguished sub-algebra $\mathfrak{u}(1)_{\rm L} \oplus \mathfrak{u}(1)_{\rm R}$; we denote the $\mathfrak{u}(1)_{\rm L} \oplus \mathfrak{u}(1)_{\rm R}$ of $\mathcal{O}$ by $q[\mathcal{O}]$ and $\overline{q}[\mathcal{O}]$. These charges are related to q^α and $\overline{q}_B$ by

$$q = t_\alpha q^\alpha \ , \qquad\qquad \overline{q} = \overline{q}_B + q \ . \tag{3.5.28}$$

The assignments for the superfields of the LG theory are therefore

(3.5.29)

	θ	Φ^a	Γ^A
$\mathfrak{u}(1)_{\rm L}$	0	$t_\alpha q^\alpha_a$	$t_\alpha Q^\alpha_A$
$\mathfrak{u}(1)_{\rm R}$	1	q_a	$1 + Q_A$

To simplify notation for what follows we set

$$q_a = q[\Phi_a] = t_\alpha q^\alpha_a \ , \qquad\qquad Q_A = q[\Gamma^A] = t_\alpha Q^\alpha_A \ , \tag{3.5.30}$$

for the $\mathfrak{u}(1)_{\rm L}$ charges of the fundamental fields.

We saw in section 3.3 that the charges q^α are realized by the KMV currents $\mathcal{K}^\alpha_\chi$ in the chiral algebra of the LG theory, with $\mathcal{T}_\chi$ serving as the Virasoro generator. By computing the OPEs of those currents with the fields, we find that ϕ_a, $\overline{\gamma}_A$ and γ^A are all KMV primary with charges and weights that follow from these symmetry assignments.

Exercise 3.6. Compute the OPEs of ϕ_a, $\overline{\gamma}_A$ and γ^A with $\mathcal{T}_\chi$ and $\mathcal{K}^\alpha_\chi$ to verify that they are indeed KMV primary, and determine their charges and weights to verify (3.6.2).

Once we know this data, we can also find the right-moving weights and $\mathfrak{u}(1)_{\rm R}$ charges $\overline{q}$. The latter follows easily since $\overline{q} = \overline{q}_B + q$, and the former is a consequence of the fact that $h - \overline{h}$, the spin of an operator is an RG invariant. Hence, we obtain the table

(3.5.31)

	ϕ_a	γ^A	$\overline{\gamma}_A$
q^α	q^α_a	Q^α_A	$-Q^\alpha_A$
q	q_a	Q_A	$-Q_A$
h	$\frac{1}{2}q_a$	$1 + \frac{1}{2}Q_A$	$-\frac{1}{2}Q_A$
$\overline{q}$	q_a	$1 + Q_A$	$-1 - Q_A$
$\overline{h}$	$\frac{q_a}{2}$	$\frac{1+Q_A}{2}$	$\frac{-1-Q_A}{2}$

Note that $\overline{\gamma}^A$ by itself does not belong to the cohomology, but we can use these assignments when $\overline{\gamma}^A$ appears in some composite $\overline{\boldsymbol{Q}}$–closed field.

We now see that the LG theory satisfies all of the assumptions of section 3.4 necessary for the existence of a B/2 topological heterotic ring. To describe that finite-dimensional space of operators, we consider the $\mathcal{O}[\omega^{k,l}]$ of (3.5.25). We can grade these by $\mathfrak{u}(1)_{\rm L} \oplus \mathfrak{u}(1)_{\rm R}$ charges simply by restricting the coefficients $\omega^{A_1\cdots A_k}_{B_1\cdots B_k}(\phi)$ to be quasi-homogeneous polynomials of definite degree

$$q\left[\omega^{A_1\cdots A_k}_{B_1\cdots B_k}(\phi)\right] = d_\omega + \sum_{s=1}^{k} Q_{A_s} - \sum_{t=1}^{l} Q_{B_t} \ . \tag{3.5.32}$$

With this choice we have

$$\begin{aligned} q\left[\mathcal{O}[\omega^{k,l}]\right] &= d_\omega \ , & \overline{q}\left[\mathcal{O}[\omega^{k,l}]\right] &= d_\omega + l - k \\ h\left[\mathcal{O}[\omega^{k,l}]\right] &= \tfrac{1}{2} q\left[\mathcal{O}[\omega^{k,l}]\right] + l \ , & \overline{h}\left[\mathcal{O}[\omega^{k,l}]\right] &= \tfrac{1}{2}\overline{q}\left[\mathcal{O}[\omega^{k,l}]\right] \ . \end{aligned} \tag{3.5.33}$$

As expected, the operators are chiral primary, but also the operators with $l = 0$ satisfy

$$h\left[\mathcal{O}[\omega^{k,0}]\right] = \tfrac{1}{2} q\left[\mathcal{O}[\omega^{k,0}]\right] \ , \tag{3.5.34}$$

which is the defining relation for operators in the B/2 topological heterotic ring. In describing the operators we have so far neglected including any holomoprhic derivatives of the fields, such as $\partial\phi_a$ and $\partial\gamma^B$; we now see that is sufficient to completely describe the B/2 ring because an operator with a holomorphic derivative will have $h > q/2$.

These observations imply that our plunge into Koszul cohomology in the previous section has not been in vain. The B/2 ring of the LG theory consists of operators $\mathcal{O}[\omega^k]$, where

$$\mathcal{O}[\omega^k] = \omega^{A_1\cdots A_k}(\phi)\overline{\gamma}_{A_1}\cdots\overline{\gamma}_{A_k} \ , \tag{3.5.35}$$

and $\omega \in H_k(K_\bullet, J_\llcorner)$. We also see that there are no operator ordering ambiguities for these observables or their OPEs; moreover, the OPE product in the B/2 ring is simply the wedge product on the forms.

Recall that we also wish to restrict attention to compact LG theories, i.e. those where the locus of simultaneous vanishing of the J_A is a collection of points. In that case the dimension of the ideal $\boldsymbol{J} = \langle J_1, \ldots, J_N \rangle$ is zero, and therefore (3.5.22) reduces to

$$H_k(K_\bullet, J_\llcorner) = \begin{cases} 0 & \text{for} \quad k > N - n \\ \text{non-zero} & \text{for} \quad k \leq N - n \ . \end{cases} \tag{3.5.36}$$

This gives a bound on the largest k (or, equivalently, smallest $\overline{q}_0$ charge) at which the B/2 ring is non-trivial.

3.6 B/2 correlation functions via localization

In general the chiral algebra leads to a scale-invariant OPE, and we may expect an interesting structure to be contained in the correlation functions

$$\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_k(z_k) \rangle \ . \tag{3.6.1}$$

A moment's thought shows that these correlation functions are, after all, not so interesting: the $\mathcal{O}_i$ must have $\overline{q} \geq 0$, so as long as at least one operator has non-zero $\mathfrak{u}(1)_{\mathrm{R}}$ charge, this correlation function will vanish.

In order to obtain a non-zero answer for correlation functions we can perform a twist of the theory. This notion, introduced in [66] in four dimensions, and in [67] in two dimensions, led to a revolution in quantum field theory and especially in its connections to mathematics. There are many reviews of the subject, for instance in [39, 68]. The key ideas relevant for our discussion are presented clearly and concisely in [69]. We will illustrate them in the LG setting.

The starting point goes back to the basic notions from chapter 1: upon continuation to Euclidean space the Lorentz group simply becomes the rotation symmetry $\mathfrak{u}(1)_\Lambda$, and its action on any field is determined by the field's spin. Any LG theory also has the $\mathfrak{u}(1)_{\mathrm{B}}$ symmetry, and we therefore have the following assignments for the basic fields:

	ϕ_a	$\overline{\phi}_a$	ψ_a	$\overline{\psi}_a$	γ^A	$\overline{\gamma}_A$
$\mathfrak{u}(1)_\Lambda$	0	0	$-1/2$	$-1/2$	$1/2$	$1/2$
$\mathfrak{u}(1)_B$	0	0	-1	1	1	-1
$\mathfrak{u}(1)_{\Lambda'}$	0	0	-1	0	1	0

(3.6.2)

The last row in the table is just a linear combination of the first two, i.e. it corresponds to the twisted Lorentz current $\mathcal{J}^\mu_{\Lambda'}$ defined by (here μ is a worldsheet vector index)

$$\mathcal{J}^\mu_{\Lambda'} = \mathcal{J}^\mu_\Lambda + \tfrac{1}{2}\mathcal{J}^\mu_{\mathrm{B}} \ . \tag{3.6.3}$$

Up to choice of sign this is the unique combination of Lorentz and $\mathfrak{u}(1)_{\mathrm{B}}$ symmetries that assigns charge 0 to the supercharge $\overline{\boldsymbol{Q}}$.

The half-twisted theory is obtained by declaring $\mathcal{J}^\mu_{\Lambda'}$ to be the Lorentz current, or equivalently, we think of putting the theory on a curved worldsheet by turning on a $\mathfrak{u}(1)_{\mathrm{B}}$ background gauge field proportional to the spin connection. From either perspective, we now have a theory equipped with a nilpotent scalar operator $\overline{\boldsymbol{Q}}$, and we can project onto its cohomology to obtain the (infinite-dimensional) spectrum of the half-twisted theory. The preceding discussion shows that we can further truncate the cohomology to the finite-dimensional B/2 ring. We call the half-twisted theory with this B/2 projection the B/2 twisted theory.[8]

[8] Since the twist and BRST operator are identical to that of the full half-twisted theory, a better but longer term might be the "B/2 heterotic ring projection of the half-twisted model."

Anticipating the BRST projection onto $\overline{\boldsymbol{Q}}$ cohomology, we rewrite the LG action as

$$\begin{aligned} S_t &= \frac{1}{2\pi}\int d^2z\,\left\{t\bar\partial\overline{\phi}^a\partial\phi^a + t\overline{\psi}^a\partial\psi^a + \overline{\gamma}_A\bar\partial\gamma^A - \gamma^A J_{A,a}\psi^a - \overline{\psi}^a\overline{J}^A_{,a}\overline{\gamma}_A + J_A\overline{J}^A\right\} \\ &= \frac{1}{2\pi}\int d^2z\,\left\{\gamma^A_z\bar\partial\overline{\gamma}_A - \gamma^A_z J_{A,a}\psi^a_{\overline{z}}\right\} + \overline{\boldsymbol{Q}}\cdot\frac{1}{2\pi}\int d^2z\,\left\{-t\psi^a_{\overline{z}}\partial\overline{\phi}^a - \overline{\gamma}_A\overline{J}^A\right\}. \end{aligned} \quad (3.6.4)$$

More precisely, we defined a family of actions depending on a parameter t; the original action is obtained at $t = 1$. We also introduced the subscripts z and $\overline{z}$ on γ^A_z and $\psi^a_{\overline{z}}$ to emphasize that on a curved worldsheet we should think of these as components of fermionic holomorphic (γ^A_z) and anti-holomorphic ($\psi^a_{\overline{z}}$) 1-forms. In that form, the action, although originally defined on a flat worldsheet, makes sense and remains $\overline{\boldsymbol{Q}}$–closed on a curved worldsheet.

The last sentence deserves a contemplative pause. We propose here that we may easily put the theory on a curved worldsheet. That in itself is perhaps not so remarkable, at least as far as the classical theory is concerned: we have a conserved energy-momentum tensor, and there should be no trouble in minimally coupling this to worldsheet gravity. The remarkable feature of the half-twisted theory is that we can do this while preserving some global supersymmetry! That is indeed a new and perhaps unexpected feature: the action will be supersymmetric for any worldsheet metric g—the supercharge simply leaves the metric invariant. The reason this works so nicely is that by twisting the theory we got rid of all of the spinors, and, more importantly, the $\overline{\boldsymbol{Q}}$ supercharge transforms as a scalar with respect to the twisted Lorentz symmetry.[9]

To make this explicit, we use coordinates $z, \overline{z}$ on the worldsheet and recall our definition $d^2z = idz \wedge d\overline{z}$, and we write $\psi^a = \psi^a_{\overline{z}}d\overline{z}$ and $\gamma^A = \gamma^A_z dz$ as our fermionic 1-forms. We can then write the following action on the curved worldsheet with a metric g:

$$S_t = \frac{i}{2\pi}\int_\Sigma \left\{\gamma^A\wedge\bar\partial\overline{\gamma}_A - \gamma^A\wedge J_{A,a}\psi^a\right\} + \overline{\boldsymbol{Q}}\cdot\int_\Sigma\left[\frac{it}{2\pi}\partial\overline{\phi}^a\wedge\psi^a - \frac{1}{\pi}*_g\overline{\gamma}_A\overline{J}^A\right], \quad (3.6.5)$$

where $*_g$ is the Hodge dual taken with respect to metric g on the worldsheet.

The action is therefore explicitly $\overline{\boldsymbol{Q}}$–closed for all t and all g; moreover, it has a $\overline{\boldsymbol{Q}}$–exact dependence on t, $\overline{J}$, and the Weyl mode of the worldsheet metric. We therefore expect correlation functions of $\overline{\boldsymbol{Q}}$–closed operators will be independent of t, $\overline{J}$ and any Weyl rescaling of the worldsheet metric. The latter in particular includes a rescaling of the worldsheet volume.

The first term in the action has a holomorphic dependence on the coefficients of monomials in the J_A. In addition, since it also explicitly involves the Dolbeault operator $\bar\partial$, it also depends holomorphically on the worldsheet complex structure. This is in line with the structure of the chiral algebra of the theory, which defines a holomorphic CFT. The half-twisted theory allows us to define this holomorphic CFT on any Riemann surface, and we expect its correlation functions of local operators to have a holomorphic dependence on the Riemann surface complex structure, as well as on the insertions of the operators.

[9] The supercharge $\boldsymbol{Q}$ transforms as a 1-form under the twisted Lorentz symmetry, and the corresponding invariance must be promoted to a local one on a curved worldsheet.

Do we expect these properties to hold in the quantum theory? This is certainly not obvious. In order to define the path integral some regularization must be introduced, and this will inevitably violate the chiral structure of the classical half-twisted theory.[10] We also know that in general our theory will suffer from global and, if $c \neq \overline{c}$, local gravitational anomalies, and these will have to be reflected in the half-twisted theory. As emphasized in [55, 70], these issues are subtle in half-twisted non-linear sigma models, where global properties of the targetspace lead to obstructions to quantum conformal invariance. We suspect that these issues do not arise in the regularization of the super-renormalizable LG Lagrangian, but the author is not aware of a complete treatment of these subtleties. However, they will play an interesting role for Riemann surfaces of genus 1 or higher, where the worldsheet metric has a non-trivial complex structure moduli space.

We will restrict attention to computations of correlation functions in the B/2 theory where the worldsheet is taken to be a sphere $\mathbb{P}^1$, and we will assume that we can regulate the theory while preserving $\overline{\boldsymbol{Q}}$–invariance. This means that the B/2 theory correlation functions will be very simple: on one hand, they have a meromorphic dependence on the operator insertions, but on the other hand, since they also have a non-singular OPE, each such correlation function must be a holomorphic function on the sphere, in other words, a constant. This is the origin of the term "topological heterotic ring" [56]—the half-twisted theory, when restricted to B/2 observables on the sphere yields correlation functions that are constants.[11]

We can compute these B/2 correlation functions on $\mathbb{P}^1$ by supersymmetric localization. Localization computations have now been carried out in diverse dimensions with various amounts of supersymmetry, and there is an extensive literature on the subject. See, for instance, [71] for a review. We will content ourselves with a more heuristic treatment here.

At its heart, supersymmetric localization relies on a beautifully simple idea [69]. Consider an integral over some domain $\mathcal{X}$ of the form

$$Z = \int_{\mathcal{X}} d\phi \; e^{-S[\phi]} \;, \tag{3.6.6}$$

and suppose $S[\phi]$ and the measure are invariant under the action of a bosonic symmetry group G. If G acts without fixed points on $\mathcal{X}$, then we can reduce the integration to the (perhaps) simpler integral over a smaller domain

$$Z = \text{Vol}(G) \int_{\mathcal{X}/G} d\phi e^{-S[\phi]} \;. \tag{3.6.7}$$

If the action of G on $\mathcal{X}$ has fixed points, we need to treat their neighborhoods more carefully. Now suppose, instead, that our symmetry is fermionic. In that case, $\text{Vol}(G) = 0$, and field

[10] The inevitability is easy to understand — a regularization introduces a lengthscale, and that choice is not compatible with the classical scale invariance.

[11] We emphasize that considerations at higher genus suggest that the theory, even when restricted to the B/2 observables, is not topological—it will depend on the worldsheet metric's complex structure and is best thought of as a holomorphic CFT.

configurations that are not fixed by the symmetry do not contribute to the integral. This means that the integral will localize onto the fixed points of the G action on $\mathcal{X}$. If we can arrange things so that the fixed point set is simple, for instance a set of isolated points, then we can often perform the integral by a saddle-point method. In supersymmetric field theory localization is particularly powerful when it can reduce a path integral computation to a finite-dimensional integral; in that happy situation we can use a path integral to obtain exact results.

Let us now see how these general ideas are manifested in the example of B/2 half-twisted LG theory. The path integral we wish to compute is

$$\langle \mathcal{O}[\omega^{k_1}](z_1) \cdots \mathcal{O}[\omega^{k_n}(z_n)\rangle = \mathcal{N} \int d[\text{fields}] e^{-S_t} \mathcal{O}[\omega^{k_1}](z_1) \cdots \mathcal{O}[\omega^{k_n}(z_n)\ . \tag{3.6.8}$$

Here $\mathcal{N}$ is a normalization constant, S_t is our t–deformed action, and $d[\text{fields}]$ is the measure. The latter is of course the source of all the subtleties in the computation. In theories with a complicated field space, like non-linear sigma models, the path integral is defined patch by patch in the targetspace. A global definition of the measure is required, and there can be global obstructions to the existence of a measure with particular properties (say targetspace diffeomorphism covariance); similar but simpler issues are faced in gauge theories, where they provide the path integral interpretation of anomalous global symmetries. These issues do not arise in the half-twisted LG theory—the targetspace has a trivial topology, and the UV definition of the theory allows us to essentially work with free fields.

The two-dimensional Laplacian and a measure

As we will now show, we can be quite concrete about the regularization of the measure for the LG theory. Let g be a generic Kähler metric on the worldsheet $\mathbb{P}^1$, and let ∇_g be the Laplacian operator constructed from this metric.

3.7 Redefinitions, parameters, and accidents

%% Singular locus discussion like in the residue paper.

Chapter 4

Heterotic non-linear sigma models

Abstract

BLAH BLAH GEOMETRY BLAH BLAH

4.1 A review of complex geometry

Let M be a compact manifold and denote by $\Omega^k(M)$ the degree k real-valued differential forms on M. The de Rham differential d is a real operator taking k-forms to $(k+1)$-forms in the usual way; in local coordinates we have

$$\omega = \tfrac{1}{k!}\omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} , \qquad d\omega = \tfrac{1}{k!}\omega_{i_1\cdots i_k,j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} . \tag{4.1.1}$$

If M is equipped with a Riemannian metric, then we also have a symmetric inner product on differential forms defined by the Hodge star:

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta . \tag{4.1.2}$$

This allows us to define the formal adjoint $d^\dagger : \Omega^k(M) \to \Omega^{k-1}(M)$ via $\langle d^\dagger \alpha, \beta \rangle = \langle \alpha, d\beta \rangle$ for all α, β. It is then not hard to see that the Hodge-de Rham Laplacian

$$\Delta_d = dd^\dagger + d^\dagger d \tag{4.1.3}$$

is a non-negative operator. Its kernel consists of the harmonic forms, i.e. forms annihilated by both d and $d^\dagger$ (such forms are said to be closed and co-closed). It takes a bit of work to show that $\dim\ker \Delta_d < \infty$, [1] but once that is established, it follows that there exists a Green's operator Σ such that on $\Omega^k(M)$ we have $\mathbb{1} = \Pi_d + \Delta_d \Sigma$, where Π_d is the projection onto the space of harmonic forms. This is of course quite similar to the structure we just described in the SCFT, but there is no obvious candidate for the grading by q since the operator d is real.

[1] A proof may be found in, for instance, [72].

A more precise analog is to be found in complex geometry. If M is a compact complex manifold, then in each coordinate patch $\mathbb{C}^n$ we can decompose complex-valued k-differential forms as $\Omega^k(M,\mathbb{C}) = \bigoplus_{p,q} \Omega^{p,q}(M,\mathbb{C})$, where in local coordinates $\omega \in \Omega^{p,q}(M,\mathbb{C})$ has the form

$$\omega = \tfrac{1}{p!}\tfrac{1}{q!}\omega_{i_1\cdots i_p\bar{\jmath}_1\cdots\bar{\jmath}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{\jmath}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\jmath}_q} \ . \tag{4.1.4}$$

Crucially, we can now decompose $d = \partial + \bar{\partial}$, where $\partial : \Omega^{p,q}(M,\mathbb{C}) \to \Omega^{p+1,q}(M,\mathbb{C})$ and $\bar{\partial} : \Omega^{p,q}(M,\mathbb{C}) \to \Omega^{p,q+1}(M,\mathbb{C})$.[2]

The de Rham differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies $d^2 = 0$ and can be used to define the cohomology groups

$$H^k_{\mathrm{dR}}(M,\mathbb{R}) = \frac{\ker d}{\operatorname{im} d} \ . \tag{4.1.5}$$

[2]Indeed, the existence of an integrable complex structure on M is equivalent to having this sort of decomposition.

Appendix A

Conformalities

A.1 Virasoro algebra

In our review of basic CFT properties we stated a number of well-known facts. For completeness we now show how these may be deduced from our axioms.

1. *The vacuum state* $|0\rangle$ *is primary and has* $(h, \overline{h}) = (0, 0)$. It corresponds to the identity operator. There is nothing to prove here: we assume the existence of an $\mathfrak{sl}_2\mathbb{C}$-invariant vacuum.

2. *The central charge* c *is positive.* Consider the state $|T\rangle = L_{-2}|0\rangle$ that corresponds $T(z)$. Using the Virasoro algebra we find the norm
$$\||T\rangle\|^2 = \langle 0|[L_2, L_{-2}]|0\rangle = \frac{c}{2}\||0\rangle\|^2 \ .$$
It follows that $c > 0$ in a unitary theory with $T(z) \neq 0$.

3. L_0 *has a non-negative spectrum.* This is obvious for any quasi-primary state $|\Phi\rangle$ with $L_0|\Phi\rangle = h|\Phi\rangle$, since
$$\|L_{-1}|\Phi\rangle\|^2 = \langle\Phi|[L_1, L_{-1}]|\Phi\rangle = 2h\||\Phi\rangle\|^2 \ .$$
By assumption any state is a sum of quasi-primary states and their descendants, which are realized by repeated action of L_{-1} (or derivatives on the operators). Since the weight of a descendant is strictly larger than that of its quasi-primary ancestor, the result follows.

4. *Every state is a sum of primary states and their Virasoro descendants.* The argument follows the familiar one for finite-dimensional representations of a simple Lie algebra. For any primary state $|\Phi\rangle$ and an ordered partition of K $[K] = (k_1, k_2, \cdots, k_p)$, with $k_i \geq k_{i+1}$, we can define the descendant
$$L_{[K]} = L_{-k_1}L_{-k_2}\cdots L_{-k_p}|\Phi\rangle \ .$$

Organize the states in the theory according to level, with level K states having weight $h = K + \epsilon$ for $0 \leq \epsilon < 1$. Unitarity implies that level $K = 0$ states are primary. Suppose every level K state is a sum of primary states and descendants, and let $|\Phi\rangle$ be a level $K+1$ state that is orthogonal to all descendant states. For any $m \in \mathbb{Z}_{>0}$ define

$$|\Psi\rangle = L_m|\Phi\rangle \ .$$

This is a level $K+1-m$ state, and is therefore a linear combination of primaries and their descendants, so that in particular $L_{-m}|\Psi\rangle$ is a descendant state, and

$$\langle\Psi|\Psi\rangle = \langle\Phi|L_{-m}\Psi\rangle = 0 \ .$$

So, unitarity shows that $|\Psi\rangle = 0$, and therefore $|\Phi\rangle$ must be primary. The result now follows by induction on K.

5. *Any operator is anti-holormophic if and only if it has weight $h = 0$.* This follows from unitarity and

$$2h\||\Phi\rangle\|^2 + \|L_1|\Phi\rangle\|^2 = \|L_{-1}|\Phi\rangle\|^2 = \||\partial\Phi\rangle\|^2 \ .$$

Of course if $h = \overline{h} = 0$ then the operator must be position-independent, and therefore (in any local quantum field theory) a constant multiple of the identity.

A.2 Superconformal current algebras

In this section we review some basic facts on N=1 superconformal current algebras (SCCAs) following [10,29,73]. We suppose that our CFT has a left-moving supercurrent G and a SCA

$$\begin{aligned}
G(z_1)G(z_2) &\sim \frac{2c/3}{z_{12}^3} + \frac{2T(z_2)}{z_{12}}, \\
T(z_1)G(z_2) &\sim \frac{3/2G(z_2)}{z_{12}^2} + \frac{\partial G(z_2)}{z_{12}} \ , \\
T(z_1)T(z_2) &\sim \frac{c/2}{z_{12}^2} + \frac{2T(z_2)}{z_{12}^2} + \frac{\partial T(z_2)}{z_{12}} \ ,
\end{aligned} \tag{A.2.1}$$

$$\begin{aligned}
\{G_r, G_s\} &= 2L_{r+s} + \tfrac{c}{12}(4r^2-1)\delta_{r+s,0}, \qquad [L_m, G_r] = \tfrac{m-2r}{2}G_{m+r}, \\
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}(m^3-m)\delta_{m+n,0}.
\end{aligned} \tag{A.2.2}$$

The vacuum is annihilated by $G_{\pm 1/2}$ and $L_{0,\pm 1}$.

Superconformal multiplets

A pair of Virasoro primary operators Ψ, Φ, with weights h and $h+1/2$ constitute a primary representation of $N=1$ iff

$$G(z)\Psi(0) \sim \frac{\Phi(0)}{z}, \qquad \text{and} \qquad G(z)\Phi(0) \sim \frac{2h\Psi(0)}{z^2} + \frac{\partial\Psi(0)}{z}. \tag{A.2.3}$$

We will assume that Ψ is fermionic, while Φ is bosonic. If we mode expand Ψ and Φ as

$$\Psi = \sum_{r\in\mathbb{Z}-h} \Psi_r z^{-r-h}, \qquad \Phi = \sum_{m\in\mathbb{Z}-h-1/2} \Phi_m z^{-m-h-1/2}, \tag{A.2.4}$$

we find the following commutators from the OPEs:

$$\begin{aligned} [L_n, \Psi_r] &= (n(h-1)-r)\Psi_{r+n}, & [L_n, \Phi_m] &= (n(h-1/2)-m)\Phi_{n+m}, \\ \{G_r, \Psi_s\} &= \Phi_{r+s}, & [G_r, \Phi_m] &= ((2h-1)r-m)\Psi_{r+m}. \end{aligned} \tag{A.2.5}$$

Let us now specialize to the real case of interest, where $h=1/2$, and $\Psi = \psi^a$, and $\Phi = j^a$.

Holomorphy and the assumption of a single-valued OPE constrain the OPEs to take a simple form:

$$\psi^a(z)\psi^b(0) \sim \frac{g^{ab}}{z}, \quad j^a(z)j^b(0) \sim \frac{G^{ab}}{z^2} + \frac{J^{ab}(0)}{z}, \quad j^a(z)\psi^b(0) \sim \frac{\Psi^{ab}(0)}{z}. \tag{A.2.6}$$

g, G are constants, and Ψ^{ab}, J^{ab} are operators of dimensions $1/2, 1$. We want to pack Ψ^{ab}, J^{ab} into an $N=1$ primary multiplet and to relate the constants. The OPEs are equivalent to

$$\begin{aligned} \{\psi^a_r, \psi^b_s\} = g^{ab}\delta_{r+s,0}, \quad [j^a_m, \psi^b_r] = \Psi^{ab}_{m+r}, \\ [j^a_m, j^b_n] = mG^{ab}\delta_{m+n,0} + J^{ab}_{n+m}. \end{aligned} \tag{A.2.7}$$

Note that $\Psi^{ab} = -\Psi^{ba}$ follows by Jacobi:

$$[j^a_m, \psi^b_r] + [j^b_m, \psi^a_r] = [\{G_{m-r}, \psi^a_r\}, \psi^b_r] + [\{G_{m-r}, \psi^b_r\}, \psi^a_r] + [\{\psi^a_r, \psi^b_r\}, G_{m-r}] = 0. \tag{A.2.8}$$

We will first show $g^{ab} = G^{ab}$ by using unitarity. Since the vacuum is annihilated by $G_{\pm 1/2}$, as well as all the raising modes and j_0, we have

$$G^{ab} = \langle 0|j^a_1 j^b_{-1}|0\rangle = \langle 0|\psi^a_{1/2}G_{1/2}G_{-1/2}\psi^b_{-1/2}|0\rangle = \langle 0|\psi^a_{1/2}\psi^b_{-1/2}\ 0\rangle = g^{ab}. \tag{A.2.9}$$

The Ψ^A, J^A are clearly Virasoro primary. Moreover, the Jacobi identity

$$\begin{aligned} \{F_1,[F_2,B]\} + [B,\{F_1,F_2\}] + \{F_2,[F_1,B]\} = 0, \implies \{F,[F,B]\} = [F^2,B] \\ [F_1,\{F_2,F_3\}] + [F_3,\{F_1,F_2\}] + [F_2,\{F_3,F_1\}] = 0 \implies [F,\{F,F_3\}] = [F^2,F_3] \end{aligned} \tag{A.2.10}$$

tells us

$$\{G_s, \Psi^{ab}_{m+r}\} = \{G_s, [j^a_m, \psi^b_r]\} = J^{ab}_{s+m+r}. \tag{A.2.11}$$

Here we used $g = G$ to cancel the central terms. Similarly, we can use the Jacobi identity and anti-symmetry of Ψ^{ab} to show that

$$[G_s, J^{ab}_m] = -m\Psi^{ab}_{m+s}. \qquad \text{(A.2.12)}$$

Thus, (Ψ^{ab}, J^{ab}) do form an $h = 1/2$ $N = 1$ primary multiplet, and as expected the algebra of the superconformal currents closes. Unitarity of the Kac-Moody algebra identifies $g^{ab} = \widehat{k}\delta^{ab}$ and leads to

$$\begin{aligned} \{\psi^a_r, \psi^b_s\} = \widehat{k}\delta^{ab}\delta_{r+s,0}, \quad [j^a_m, \psi^b_r] = if^{abc}\psi^c_{m+r}, \\ [j^a_m, j^b_n] = m\widehat{k}\delta^{ab}\delta_{m+n,0} + if^{abc}j^c_{n+m}. \end{aligned} \qquad \text{(A.2.13)}$$

There are actually two current algebras hiding in this system: let

$$\widetilde{J}^a_\psi \equiv -\frac{i}{2k} f^{abc}\psi^b\psi^c \ . \qquad \text{(A.2.14)}$$

These form a current algebra with level $h(g)$—the dual Coxeter number. Moreover, the current $\widetilde{J}^a = j^a - \widetilde{J}^a_\psi$ commutes with $\widetilde{J}^a_\psi$ and has level $k_b = k - h(g)$. A Sugawara construction for these two sets of currents leads to the total central charge

$$c_{\text{SKM}} = \left(\frac{k_b}{k} + \frac{1}{2}\right) \dim \mathfrak{g} \ . \qquad \text{(A.2.15)}$$

There is a super-Sugawara construction as well.

Currents and (1,0) SUSY

In the previous section we saw how a KM current can naturally show up as the top component of an N=1 multiplet. Must this be so? To examine the point in detail, we suppose that we have the N=1 SCA and a level k KM current J. Our first concern is the action, determined by the OPE, of $G_{\mp 1/2}$ on $J(z)$. For any operator X

$$G_r \cdot X(w) \equiv [G_r, X(w)]_\pm = \frac{1}{2\pi i}\oint_{C(w)} dz\ z^{r+1/2} G(z) X(w) \ . \qquad \text{(A.2.16)}$$

Now using $X = J$, we see that the most general holomorphic and single-valued G–J OPE

$$G(z)J(w) \sim \frac{\Psi(w)}{(z-w)^2} + \frac{\partial\Psi(w) - \mathcal{X}(w)}{z-w} \qquad \text{(A.2.17)}$$

leads to

$$G_r \cdot J(w) = \partial\left[w^{r+1/2}\Psi(w)\right] - w^{r+1/2}\mathcal{X}(w) \iff [G_r, J_n] = -n\Psi_{r+n} - \mathcal{X}_{r+n} \ , \qquad \text{(A.2.18)}$$

where we used mode expansions $\Psi(w) = \sum_r \Psi_r w^{-r-1/2}$ and $\mathcal{X}(w) = \sum_r \mathcal{X}_r w^{-r-3/2}$. Thus, we see that

$$\mathcal{X}_r = [J_0, G_r], \qquad \Psi_s = [G_{s+1}, J_{-1}] + [J_0, G_s] \ . \tag{A.2.19}$$

At least one of these is non-zero, since otherwise $G_{\pm 1/2}J_{-1}|0\rangle = 0$, an impossibility in a unitary theory:

$$0 = \langle 0|J_1 G_{1/2} G_{-1/2} J_{-1}|0\rangle = 2\langle 0|J_1 J_{-1}|0\rangle = 2k\, \| |0\rangle \|^2 \ . \tag{A.2.20}$$

It is easy to show

$$[L_m, \mathcal{X}_r] = \frac{m-2r}{2}\mathcal{X}_{m+r} \ , \qquad [L_m, \Psi_s] = \frac{-m-2s}{2}\Psi_{m+s} \ . \tag{A.2.21}$$

Hence $\Psi(z)$ and $\mathcal{X}(z)$ are Virasoro primary fields with weights 1/2 and 3/2 respectively. In fact, Ψ is also N=1 primary. To see that, it is simplest to work with the corresponding state $|\Psi\rangle \equiv \lim_{z\to 0}\Psi(z)|0\rangle = \Psi_{-1/2}|0\rangle$. This is N=1 primary iff $G_{1/2}|\Psi\rangle = 0$. We compute

$$G_{1/2}|\Psi\rangle = G_{1/2}\left[[G_{1/2}, J_{-1}] + [J_0, G_{-1/2}]\right]|0\rangle = G_{1/2}^2 J_{-1}|0\rangle = L_1 J_{-1}|0\rangle = J_0|0\rangle = 0. \tag{A.2.22}$$

The second equality follows because $G_{\pm 1/2}|0\rangle = 0$ and $J_0|0\rangle = 0$. Hence, we have

$$G(z)\Psi(w) = \frac{K(w)}{z-w} \tag{A.2.23}$$

for a weight 1 operator $K(z) \equiv G_{-1/2}\cdot\Psi(z)$ with moding

$$K(z) = \sum_n K_n z^{-n-1}, \qquad K_n \equiv \{G_{-1/2}, \Psi_{n+1/2}\} = \{G_r, \Psi_{n-r}\}. \tag{A.2.24}$$

It is a Virasoro primary field since the corresponding state

$$|K\rangle = \lim_{z\to 0} K(z)|0\rangle = \lim_{z\to 0}\left(z^{-1}\{G_{-1/2}, \Psi_{1/2}\} + \{G_{-1/2}, \Psi_{-1/2}\}\right)|0\rangle = G_{-1/2}\Psi_{-1/2}|0\rangle \tag{A.2.25}$$

satisfies

$$L_1|K\rangle = L_1 G_{-1/2}\Psi_{-1/2}|0\rangle = G_{1/2}|\Psi\rangle = 0 \ . \tag{A.2.26}$$

This is a non-trivial object as long as $|\Psi\rangle \neq 0$:

$$\| |K\rangle \|^2 = \langle 0|\Psi_{1/2} G_{1/2} G_{-1/2}\Psi_{-1/2}|0\rangle = \| |\Psi\rangle \|^2 \ . \tag{A.2.27}$$

Moreover, from $G_{1/2}K_{-1}|0\rangle = \Psi_{-1/2}|0\rangle$ and $G_{-1/2}K_{-1}|0\rangle = L_{-1}\Psi_{-1/2}|0\rangle$, we also see

$$G(z)K(w) = \frac{\Psi(w)}{(z-w)^2} + \frac{\partial\Psi(w)}{z-w} \iff [G_r, K_m] = -m\Psi_{r+m} \ . \tag{A.2.28}$$

This is sufficient to show that (Ψ, K) form an N=1 superconformal multiplet, and in fact a component of a superconformal current algebra explored in the previous section.

The remaining structure is reasonably clear. Given the full set of currents $\{J^A\}$, we construct the SCCA with multiplets (Ψ^α, K^α) for each α such that $|\Psi^\alpha\rangle \neq 0$. We can then choose a basis such that the remaining currents, labeled by R^a, will commute with Ψ^α, K^α and will act on the supercharges:

$$R^a(z)G(w) = \frac{\mathcal{X}^a(w)}{z-w} \iff [R^a_n, G_r] = \mathcal{X}^a_{n+r} \ . \tag{A.2.29}$$

We can now apply the classification of superconformal algebras referenced in the text to see that the R^a must generate the corresponding R-symmetry algebra.

Appendix B

Elements of geometry

In this chapter we provide a discussion of various geometric notions relevant for us. We have three aims here. First, we will establish our notation. Second, we will remind the reader of some basic notions with which she is probably familiar. Finally, we will attempt to introduce the more advanced material sufficiently so that the reader has a sense for its structure.

We assume that the reader is reasonably familiar with standard differential geometry notions of differential manifolds and Riemannian geometry. There are many excellent references geared towards physicists of much of this material, for instance the early review [74], as well as [75]. Aspects of complex/Hermitian/Kähler/toric geometry are perhaps a little bit less familiar, but fortunately they are also presented in a number of excellent texts (though these are mostly geared towards mathematicians). A list of geometric references that the author found particularly useful is the following.

1. The introductory chapters of Griffiths and Harris [72] contain fundamental notions of Hermitian and Kähler geometries and their relation to algebraic geometry.

2. Bott and Tu [76] provides a very readable introduction to differential topology and characteristic classes, and these notions are explored further in the context of symplectic geometry in %McDuffSalamon. Rudiments of symplectic geometry are to be found in the timeless Arnold %Arnold, and a very elegant introduction to notions classic and modern is in %Bryant.

3. Kodaira [77] is very readable discussion of deformation theory and is grounded in differential geometry.

4. Voisin gives a comprehensive introduction to complex geometry and Hodge theory in [78]; an excellent shorter set of notes is to be found in [79]. A very readable discussion of the $\partial\bar{\partial}$-lemma and related geometric concepts is in the thesis of Angella.

5. Special and exceptional holonomy manifolds are described in the book by Joyce [80].

6. The geometry of gauge theory is presented in many texts, including the classic works of Freed and Uhlenbeck [81]. The classic paper of Atiyah et al [82] is quite readable and useful.

7. Aspects of Kähler geometry is discussed in many reviews, e.g. [83, 84].

8. A review of Calabi-Yau geometry and its relation to (2,2) SCFT is given in [39]; a very readable introduction especially geared towards physicists is given by Aspinwall in [85].

9. The geometry complex surfaces is discussed in the classic text of Barth et al [86]. K3 surfaces in particular are discussed in many references including Aspinwall's lecture notes [87] and Huybrecht's lecture notes [88].

10. There is an infinite number of sources on toric geometry. There is now a quite comprehensive and modern treatment given in by Cox et al in [89]. The classic introductory treatment given by Fulton [90] is readable but does not quite provide all the tools. The toric geometry chapters of the Mirror Symmetry book by Cox and Katz [91] is excellent.

B.1 Real geometry

Let M be a smooth manifold of dimension n. When M is real n is the real dimension; when M is complex, we will most often talk about its complex dimension, the two will be denoted by $\dim_{\mathbb{R}} M$ and $\dim_{\mathbb{C}} M$ respectively when there is a possibility of confusion. They are related by $\dim_{\mathbb{R}} M = 2\dim_{\mathbb{C}} M$. In this section we focus on the real case. In what follows we will use the summation convention unless otherwise noted.

First steps

Given M we construct its tangent bundle T_M. Its sections are the vector fields on M, which in a local patch $U \simeq \mathbb{R}^n$ with coordinates x^μ take the form

$$V = V^\mu(x)\frac{\partial}{\partial x^\mu} \ . \tag{B.1.1}$$

The cotangent bundle T_M^* is dual to T_M, and its sections are the differential 1-forms on M, which in local coordinates take the form

$$\omega = \omega_\mu(x) dx^\mu \ . \tag{B.1.2}$$

In this basis duality is expressed through the relation

$$dx^\mu(\frac{\partial}{\partial x^\nu}) = \delta^\mu_\nu \ , \tag{B.1.3}$$

where δ^μ_ν is the Kronecker delta. More general tensors are sections of $T_M^{\otimes k} \otimes (T_M^*)^l$, and a particularly important class is the set of k-forms, which are sections of the bundle $\wedge^k T_M^*$.

Vector bundles and their generalizations

All of these are examples of vector bundles over M. A vector bundle $V \to M$ can be thought of as a family of vector spaces $E_x \simeq \mathbb{R}^k$ that are labeled by points $x \in M$ and fit together in a way that is compatible with the differentiable structure on M and the linear structure on the vector space.

In order to set our notation for these structures, we will describe the basic elements in their construction. A rank k vector bundle $\pi : V \to M$ is a differentiable manifold with a projection map $\pi : V \to M$ such that the inverse image of any point $x \in M$ is $\pi^{-1}(x) = V_x \simeq \mathbb{R}^k$, and every x is contained in an open neighborhood U, such that there is a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and for every $y \in U$ $\pi^{-1}(V_y) = \{y\} \times \mathbb{R}^k$. As explained in, for instance, [72], every vector bundle can equivalently be thought of as an open cover $\{U_\alpha\}$ of M and a set of transition functions $g_{\alpha\beta}$ defined on each non-empty overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and satisfying the following properties:

$$g_{\alpha\beta}(x) \in \mathrm{GL}(k, \mathbb{R}) \ , \qquad g_{\alpha\beta}(x) g_{\beta\alpha}(x) = \mathbb{1} \ , \qquad g_{\alpha\beta}(x) g_{\beta\gamma}(x) g_{\gamma\alpha}(x) = \mathbb{1} \ . \tag{B.1.4}$$

The last requirement must hold for all non-empty triple-overlaps. The $g_{\alpha\beta}$ tell us how to identify the fiber vector spaces on the overlaps.

A section s of the bundle $V \to M$ over an open set $X \subset M$ is a map $s : X \to V$ such that $s(x) \in V_x$ for all $x \in U$. In terms of the open cover $\{U'_\alpha\}$ with $U'_\alpha = U_\alpha \cap X$ we may think of s as a collection of s_α constrained by the transition functions: for all non-empty overlaps $U'_{\alpha\beta}$ we have

$$s_\alpha = g_{\alpha\beta} s_\beta \ . \tag{B.1.5}$$

We will use the notation $\Gamma(X, V)$ to denote the space of sections of the bundle over a space X. This is clearly a vector space, since given two sections $s^{(1)}$ and $s^{(2)}$, their sum will also be a section. Every vector bundle $V \to M$ has a trivial global section: $s_\alpha = 0$.

Vector bundles inherit operations familiar from linear algebra. For instance, $V_1 \to M$ and $V_2 \to M$ are vector bundles of rank, respectively, k_1 and k_2, then we can construct vector bundles $V_1 \oplus V_2 \to M$ and $V_1 \otimes V_2 \to M$ of rank, respectively, $k_1 + k_2$ and $k_1 k_2$ by taking a direct sum or product of the transition functions.

This construction has a myriad of generalizations and special cases. For us, an important example of a specialization is the class of complex vector bundles over a complex manifold. An important example of a generalization is given by more general fiber bundles, where the fiber, instead of being a vector space, is taken to be a manifold with more complicated structure and perhaps topology.

Conventions for differential forms

Differential forms are essential tools of differential geometry and topology. In this section we review our conventions for these objects. We set $\Omega^k(M) = \Gamma(M, \wedge^k T^*_M)$—this is the space of differential forms of degree k on M. In terms of local coordinates we normalize the coefficients as

$$\omega = \frac{1}{k!}\omega_{\mu_1\mu_2\cdots\mu_k}(x)dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_k} \ . \tag{B.1.6}$$

We will frequently suppress the wedge symbol if there is no possibility of confusion, i.e. $dx^1 \wedge dx^2$ will be abbreviated as dx^1dx^2; unless otherwise noted, we will use the Einstein convention that repeated indices are to be summed.

The degree k runs from 0 to n, the dimension of M. The degree 0 forms are functions on M, while the degree n forms are densities that may be integrated over the manifold. We will work with orientable manifolds, where the notion of the integrable densities and top forms coincide.

The wedge product of two forms, $\wedge : \Omega^{k_1}(M) \times \Omega^{k_2}(M) \to \Omega^{k_1+k_2}(M)$ gives the vector space $\bigoplus_k \Omega^k(M)$ the structure of a graded algebra. It is graded since

$$\omega_1 \wedge \omega_2 = (-1)^{k_1k_2}\omega_2 \wedge \omega_1 \ , \tag{B.1.7}$$

where k_i is the degree of ω_i.

Another key notion is the de Rham differential operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$. This is a linear nilpotent operator, and in local coordinates its action is as follows:

$$d\omega = \frac{1}{k!}\partial_{\mu_0}\omega_{\mu_1\mu_2\cdots\mu_k}dx^{\mu_0} \wedge dx^{\mu_1} \cdots \wedge dx^{\mu_k} \ . \tag{B.1.8}$$

We say that a differential form ω is closed if $d\omega = 0$; if we can find a form η such that $\omega = d\eta$, then we say ω is exact. Because $d^2 = 0$ every exact form is closed, and the cohomology groups

$$H^k(M,\mathbb{R}) = \frac{\ker\{d : \Omega^k(M,\mathbb{R}) \to \Omega^{k+1}(M,\mathbb{R})\}}{\mathrm{im}\{d : \Omega^{k-1}(M,\mathbb{R}) \to \Omega^k(M,\mathbb{R})\}} \ . \tag{B.1.9}$$

characterize the space of closed forms modulo exact forms. A closed form $\omega \in \Omega^k(M,\mathbb{R})$ defines an equivalence class $[\omega] \in H^k(M,\mathbb{R})$ and any two representatives of the same equivalence class differ by an exact form. That is, if $[\omega] = [\omega']$ then $\omega' = \omega + d\eta$ for some $\eta \in \Omega^{k-1}(M,\mathbb{R})$.

Exercise B.1. If the reader is not familiar with this, it is a good exercise to check the product rule

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge d\omega_2 \ . \tag{B.1.10}$$

This is much like the familiar Leibniz rule except for the extra degree-dependent sign; for this reason one sometimes sees the terminology that d is an "antiderivation."

The reader should also verify the crucial property

$$d^2 = 0 \tag{B.1.11}$$

and check that the wedge product gives a graded algebra structure to the vector space $\bigoplus_k H^k(M,\mathbb{R})$. That is, if ω_1 is a representative of a class in $H^{k_1}(M,\mathbb{R})$ and ω_2 represents a class in $H^{k_2}(M,\mathbb{R})$, then $[\omega_1 \wedge \omega_2] \in H^{k_1+k_2}(M,\mathbb{R})$, and this class is independent of the choice of representatives.

Differential forms are dual to tangent vectors. A differential 1-form gives a linear map $T_M \to \Omega^0(M)$: a vector $v^\mu \frac{\partial}{\partial x^\mu}$ is sent to $v^\mu \omega_\mu$; moreover, every linear map $T_M \to \Omega^0(M)$ is obtained in this fashion. All of this extends to degree k forms, which are dual to $\wedge^k T_M$. Taking the dual of the dual, we can reverse this correspondence and think of $\wedge^k T_M$ as the dual of $\Omega^k(M)$.

This structure motivates the contraction of a vector into a differential form. Given a vector field v and a degree k form ω_k, we define the contraction as follows:

$$\begin{aligned} v\llcorner : \Omega^k(M) &\to \Omega^{k-1}(M) \\ v\llcorner\omega &= \frac{1}{(k-1)!} v^{\mu_0}\omega_{\mu_0\mu_1\cdots\mu_{k-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{k-1}} \ . \end{aligned} \tag{B.1.12}$$

More generally, given a degree $m < k$ multi-vector $v \in \wedge^m T_M$, we set the contraction (also sometimes called the interior product) to be

$$\begin{aligned} v\llcorner : \Omega^k(M) &\to \Omega^{k-m}(M) \\ v\llcorner\omega &= \frac{1}{m!}\frac{1}{(k-m)!} v^{\mu_1\cdots\mu_m}\omega_{\mu_1\cdots\mu_m\nu_1\cdots\nu_{k-m}} dx^{\nu_1} \cdots dx^{\nu_{k-m}} \ . \end{aligned} \tag{B.1.13}$$

The Hodge star

So far nothing we said depends on a choice of metric on M. Once we choose a metric, we obtain extra structure on the differential forms. The most important ingredient in this structure is the Hodge star, an invertible metric dependent linear map $*_g : \Omega^k(M) \to \Omega^{n-k}(M)$. We say that $\Omega^k(M)$ is Hodge-dual to $\Omega^{n-k}(M)$. We define it as follows. Given a metric g we construct a volume form on M as

$$\mathrm{dVol}_g = \sqrt{\det g} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \ . \tag{B.1.14}$$

Naturally, its integral yields the volume of the manifold with respect to the metric g.[1]

For any $\beta \in \Omega^k(M)$ define the Hodge star $*\beta \in \Omega^{n-k}(M)$ by the property that for any $\alpha \in \Omega^k(M)$ we have

$$\alpha \wedge *\beta = \mathrm{dVol}_g \frac{1}{k!} \alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_k} g^{i_1 j_1} \cdots g^{i_k j_k} \ , \tag{B.1.15}$$

where g^{ij} denotes the inverse of the metric: $g^{ij}g_{jk} = \delta^i_k$.

Exercise B.2. Show that the following definition works:

$$*_g\beta = \frac{\sqrt{\det g}}{k!(n-k)!} \beta^{i_1\cdots i_k} \epsilon_{i_1\cdots i_k j_1\cdots j_{n-k}} dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}} \ . \tag{B.1.16}$$

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[1] M must be orientable in order for this construction to make sense, and in writing dVol_g we made a choice of the orientation.

Here the indices on β on the right-hand-side are raised with the inverse metric, and $\epsilon_{i_1\cdots i_n}$ is fully antisymmetric and normalized to $\epsilon_{12\cdots n} = +1$. Show also that

$$*_g^2 \Omega^k(M) = (-1)^{k(n-k)} \Omega^k(M) \ . \tag{B.1.17}$$

The diligent reader may pause here to show that $*_g \beta$ is invariant under changes of coordinates on M; while performing that check one may as well check that $d\beta$ is also invariant.

Note that we will omit the g subscript on $*$ when this is unlikely to cause confusion.

Evidently, $*$ exists and is a symmetric operation: $\alpha \wedge *\beta = \beta \wedge *\alpha$; moreover, on a compact manifold we can integrate this to define

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta \ , \tag{B.1.18}$$

a symmetric bilinear positive-definite form: $\langle \alpha, \alpha \rangle = 0$ if and only if $\alpha = 0$ point-wise on the manifold.

Exercise B.3. The Hodge star also interacts nicely with the contraction operation. Given a k-form η and a $k+l$ form ω, we define the multi-vector $\widetilde{\eta}$ by raising the indices on η with the inverse metric g^{ij}. We can then make a slight abuse of notation and write $\eta \llcorner \omega = \widetilde{\eta} \llcorner \omega$. That is, we define a contraction of a k-form into a $k+l$ form by converting the former into a multi-vector via the inverse metric, and then performing the contraction operation. With that definition in hand, show that the following relations hold:

$$\eta \llcorner \omega = (-1)^{k(n-k-l)} * (\eta \wedge *\omega) \qquad \Longleftrightarrow \qquad *(\eta \llcorner \omega) = (-1)^{kl} \eta \wedge *\omega \ . \tag{B.1.19}$$

The Hodge–de Rham Laplacian

The Hodge inner product on differential forms can be used to define the formal adjoint of d, $d^\dagger$.[2] So, $d^\dagger$ is a linear differential operator such that for any $\alpha \in \Omega^k$ and any $\beta \in \Omega^{k-1}(M)$ we have

$$\langle d^\dagger \alpha, \beta \rangle = \langle \alpha, d\beta \rangle \ ; \tag{B.1.20}$$

in other words, $d^\dagger$ is a linear differential operator that lowers the degree of the form:

$$d^\dagger : \Omega^k(M) \to \Omega^{k-1}(M) \tag{B.1.21}$$

We can easily write down its explicit form in terms of $*$ and d as follows. We have

$$\langle d^\dagger \alpha, \beta \rangle = \langle \beta, d^\dagger \alpha \rangle = \int_M \beta \wedge *d^\dagger \alpha \tag{B.1.22}$$

[2]The "formal" in the definition refers to the fact that we have yet to specify the functional domain on which the operators d and $d^\dagger$ are acting.

On the other hand,

$$\langle \alpha, d\beta \rangle = \langle d\beta, \alpha \rangle = \int_M (d\beta) \wedge *\alpha \ . \tag{B.1.23}$$

But, since

$$d(\beta \wedge *\alpha) = d\beta \wedge *\alpha + (-1)^{k-1} \beta \wedge d * \alpha \ , \tag{B.1.24}$$

as long as we can drop the total derivative term on the left (this is justified when M is compact without boundary), we have

$$\langle \alpha, d\beta \rangle = (-1)^k \int_M \beta \wedge d * \alpha \ , \tag{B.1.25}$$

and comparing (B.1.22) and (B.1.25), we find that

$$*d^\dagger \alpha = (-1)^k d * \alpha \ . \tag{B.1.26}$$

Finally, using $*^2 \Omega^{k-1}(M) = (-1)^{(k-1)(n-k+1)}$, we obtain that for $\alpha \in \Omega^k(M)$

$$d^\dagger \alpha = -(-1)^{n(k-1)} * d * \ . \tag{B.1.27}$$

In particular, on an even-dimensional M this simplifies to $d^\dagger \alpha = - * d*$. Note that while $d^\dagger d^\dagger = 0$, $d^\dagger$ is not an antiderivation with respect to the wedge product!

Once we have the adjoint operator, we define the Hodge-de Rham Laplacian by

$$\Delta_d = d^\dagger d + d d^\dagger \ . \tag{B.1.28}$$

Note that the operator depends on the differential structure (i.e. on d) and on the Riemannian metric g. This is a formally self-adjoint positive semi-definite operator that commutes with $*$, d, and $d^\dagger$. Its kernel consists of forms that are closed (i.e. annihilated by d) and co-closed (i.e. annihilated by $d^\dagger$). To see the latter statement, we observe

$$\langle \alpha, \Delta_d \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle d^\dagger \alpha, d^\dagger \alpha \rangle \ . \tag{B.1.29}$$

We say that a form is harmonic if and only if it is closed and co-closed. Since Δ_d is linear, the space of harmonic forms of degree k, denoted by $\mathcal{H}^k(M)$, is a vector subspace of $\Omega^k(M)$. Note that if α is harmonic, then so is $*\alpha$; however, it is in general not true that the wedge product of two harmonic forms is harmonic. Constant functions and their Hodge duals (constant multiples of the volume form) are simple examples of harmonic forms.

With some basic analysis tools for elliptic operators on compact manifolds%REF, it is possible to prove that the solutions ω to Laplace equation with C^∞ source η

$$\Delta_d \omega = \eta \tag{B.1.30}$$

are C^∞ and form a finite-dimensional vector space. In particular, the kernel of Δ_d is finite dimensional, i.e. $\dim \mathcal{H}^k < \infty$, and there is a projector onto harmonic forms Π_d and a Green's operator Σ, such that on every $\Omega^k(M)$ we have the identity

$$\mathbb{1} = \Pi_d + \Delta_d \Sigma \tag{B.1.31}$$

and a unique decomposition of any form into a sum of three orthogonal (with respect to the L^2 norm) terms: the harmonic piece, the exact piece, and the co-exact piece:

$$\omega = (\Pi_d \omega) + d(d^\dagger \Sigma \omega) + d^\dagger(d\Sigma\omega) \ . \tag{B.1.32}$$

Thus, the Laplace equation with source has solution if and only if $\Pi_d \eta = 0$.

The decomposition implies that the degree k harmonic forms are in 1:1 correspondence with elements of the k-th de Rham cohomology group $H^k(M, \mathbb{R})$ defined by

$$H^k(M,\mathbb{R}) = \frac{\ker\{d : \Omega^k(M,\mathbb{R}) \to \Omega^{k+1}(M,\mathbb{R})\}}{\operatorname{im}\{d : \Omega^{k-1}(M,\mathbb{R}) \to \Omega^k(M,\mathbb{R})\}} \ . \tag{B.1.33}$$

The proof is simple. Let ω be a representative of a class $[\omega] \in H^k(M,\mathbb{R})$. Since ω is closed it has no co-closed term in its decomposition, and therefore $\omega = \Pi_d\omega + d\eta$. But, then $\Pi_d\omega \in \mathcal{H}^k$, and $[\Pi_d\omega] = [\omega]$. In particular this shows that $H^k(M,\mathbb{R})$ is finite-dimensional.[3] We call the dimension $b_k = \dim H^k(M,\mathbb{R})$ the k-the Betti number of M. Note that the definition of $H^k(M,\mathbb{R})$ is independent of the metric on M, so that b_k are topological invariants of the manifold. From this we learn the somewhat surprising fact that $\dim \mathcal{H}^k$ is also metric-independent, even though it relied on Δ_d, an operator that depends on the choice of metric!

Exercise B.4. Use the $*$ operator and preceding results to prove that $b_k = b_{n-k}$, where $n = \dim M$.

Riemannian geometry conventions

Having fixed a metric g on M, we can also define the Levi-Civita connection. We will not bother with the axiomatic definition, as this is probably familiar, and merely state our conventions.

We will occasionally be working with a connection that has torsion, so we will first present some general results before moving on to the torsion–free case. Quite generally, we take ∇ to be a covariant derivative that acts on functions as $\nabla f = df$ and on 1-forms and vector fields as

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho \ , \qquad \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V_\rho \tag{B.1.34}$$

The action on all other tensors extends by Leibniz rule. In general, the connection Γ may have torsion: the tensor

$$T^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} \tag{B.1.35}$$

[3] There are, of course, more algebraic proofs that do not rely on elliptic theory; see, for instance, [76].

need not be zero.

Because partial derivatives commute, the commutator of two covariant derivatives is a first-order differential operator. On functions

$$[\nabla_\mu, \nabla_\nu] f = -T^\alpha_{\mu\nu} \nabla_\alpha f \ , \tag{B.1.36}$$

while on more general tensors the action involves the curvature tensor. The form of the latter can be fixed from the action of ∇ on 1-forms, since the action on other tensors will follow from Leibniz rule.

$$[\nabla_\mu, \nabla_\nu]\omega_\alpha = (R_{\mu\nu})_\alpha{}^\beta \omega_\beta - T^\beta_{\mu\nu} \nabla_\beta \omega_\alpha \ . \tag{B.1.37}$$

The curvature tensor R has components

$$(R_{\mu\nu})_\alpha^{\ \beta} = \partial_\nu \Gamma^\beta_{\mu\alpha} - \partial_\mu \Gamma^\beta_{\nu\alpha} + \Gamma^\beta_{\nu\gamma}\Gamma^\gamma_{\mu\alpha} - \Gamma^\beta_{\mu\gamma}\Gamma^\gamma_{\nu\alpha} \tag{B.1.38}$$

and satisfies the Bianchi identity

$$(R_{\mu\nu})_\alpha^{\ \beta} + (R_{\alpha\mu})_\nu^{\ \beta} + (R_{\nu\alpha})_\mu^{\ \beta} = -\nabla_\mu T^\beta_{\nu\alpha} - \nabla_\alpha T^\beta_{\mu\nu} - \nabla_\nu T^\beta_{\alpha\mu} \ . \tag{B.1.39}$$

We will often write the components of the curvature tensor as $R_{\mu\nu\alpha}{}^\beta$ and also define $R_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha}{}^\gamma g_{\gamma\beta}$.

∇ is a metric connection if the metric is covariantly constant: $\nabla g = 0$. The Levi-Civita connection is the unique torsion-free metric connection, and its components are given by

$$\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha} \right) \ . \tag{B.1.40}$$

For the Levi-Civita connection the commutator $[\nabla_\mu, \nabla_\nu]$ is an algebraic operator, and the curvature obeys the well-known identities

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= -R_{\nu\mu\alpha\beta} = R_{\alpha\beta\mu\nu} \ , \\ 0 &= R_{\mu\nu\alpha\beta} + R_{\alpha\mu\nu\beta} + R_{\nu\alpha\mu\beta} \ , \\ 0 &= \nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\nu R_{\alpha\mu\beta\gamma} + \nabla_\mu R_{\nu\alpha\beta\gamma} \ . \end{aligned} \tag{B.1.41}$$

Given any point p on the manifold we can find coordinates such that the Levi-Civita connection vanishes at p; more generally, the torsion tensor T measures the obstruction to finding coordinates that make the connection vanish at a point.

Exercise B.5. Verify the following relationships between the Levi-Civita connection and the Hodge-de Rham Laplacian, where f is a function η is a $p+1$-form, and ω is a 1-form:

$$\begin{aligned} (d^\dagger \eta)_{\mu_1 \cdots \mu_p} &= -\nabla^{\mu_0} \eta_{\mu_0 \mu_1 \cdots \mu_p} \ , \\ \nabla_d f &= -\nabla^2 f = g^{\mu\nu} \nabla_\mu \nabla_\nu f \ , \\ (\nabla_d \omega)_\mu &= -\nabla^2 \omega_\mu + [\nabla_\nu, \nabla_\mu] \omega^\nu \ . \end{aligned} \tag{B.1.42}$$

B.2 Complex geometry

We begin our discussion of complex geometry with a coarser notion of an almost complex structure. A manifold M admits and almost complex structure if and only if there is a map $J : T_M \to T_M$ that satisfies $J^2 = -\mathbb{1}$, or in components $J^\mu_\nu J^\nu_\rho = -\delta^\mu_\rho$. There are global topological restrictions on the manifolds that admit an almost complex structure %McDuff-Salamon; for instance, the sphere S^4 does not admit an almost complex structure, while S^6 does.

The notion of an almost complex (AC) structure is equivalent to that of an almost symplectic (AS) structure. We say M is almost symplectic if and only if it admits a non-degenerate two-form $\omega \in H^2(M, \mathbb{R})$.

Exercise B.6. Prove that an almost complex structure (AC) is equivalent to an almost symplectic structure (AS).%McDuffSalamon. Fix a metric g on M. Supposing that M has an AC structure J, show that

$$\omega_{\mu\nu} = \tfrac{1}{2}(J^\rho_\mu g_{\rho\nu} - J^\rho_\nu g_{\rho\mu})$$

is non-degenerate. [Hint: show that $\omega(v, J(v))$ is non-zero if v is non-zero.] This shows that $\dim M$ is even, since an odd-dimensional antisymmetric matrix necessarily has a zero eigenvalue.

Next, supposing that ω is an AS structure on M, construct an AC structure. This is a little trickier. To achieve the goal recall that if A is an invertible anti-symmetric matrix, then $-A^2 = AA^T$ is a positive definite symmetric matrix with a positive square root, a symmetric matrix B such that $-A^2 = AA^T = B^2$. Show that $[A, B] = 0$ and $[A, B^{-1}] = 0$ [Hint: compute $A[A, B]A$ and conclude that $J = AB^{-1}$ satisfies $J^2 = -\mathbb{1}$. Now apply these linear algebra results to $A^\rho_\mu = \omega_{\mu\lambda} g^{\lambda\rho}$.

Let M have dimension $2n$ and let us suppose it admits an AC (and therefore an AS) structure. The AC and AS structures have four important refinements. The most basic of these is an almost-Hermitian structure: this is a choice of J and metric g such that $\omega_{\mu\lambda} = J^\rho_\mu g_{\rho\lambda}$ is anti-symmetric. The remaining ones are a complex, symplectic, and Kähler structures.

Symplectic manifolds

M admits a symplectic structure if there exists a closed non-degenerate $\omega \in \Omega^2(M)$. The local structure of symplectic geometry is somewhat trivial: the Darboux theorem asserts that in any sufficiently small open neighborhood we may find coordinates $\{q^1, p_1, q^2, p_2, \ldots, q^n, p_n\}$ such that

$$\omega = dq^1 \wedge dp_1 + dq^2 \wedge dp_2 + \cdots + dq^n \wedge dp_n \ . \tag{B.2.1}$$

Given an AS structure ω, the three-form $d\omega$ is the obstruction to having a symplectic structure and the existence of corresponding Darboux coordinates.

A symplectic manifold comes with an orientation and a symplectic volume form

$$\mathrm{dVol}_\omega(M) = \frac{1}{n!}\omega^n \ . \tag{B.2.2}$$

If M is compact, then the volume form cannot be exact, and therefore ω cannot be exact either; in other words, $[\omega]$ defines a non-zero class in $H^2(M,\mathbb{R})$.

Complex manifolds

There are several equivalent ways of describing a complex structure. Consider the cotangent bundle $T^*_M = \Omega^1(M)$. Its sections are the 1-forms, and we can consider these with complex coefficients as opposed to real ones. The space of all such forms is a complex vector space $\Omega^1(M,\mathbb{C})$. If M has an almost complex structure, then we can define the projector operators

$$\Pi^{1,0} = \frac{1}{2}(\mathbb{1} - iJ) \ , \qquad \Pi^{0,1} = \frac{1}{2}(\mathbb{1} + iJ) \tag{B.2.3}$$

and therefore decompose the 1-forms as

$$\Omega^1(M,\mathbb{C}) = \Omega^{1,0}(M,\mathbb{C}) \oplus \Omega^{0,1}(M,\mathbb{C}) \ . \tag{B.2.4}$$

In a similar fashion, we define the projectors $\Pi^{p,q}$ on $\Omega^{p+q}(M,\mathbb{C})$, such that

$$\Omega^{p,q}(M,\mathbb{C}) = \Pi^{p,q}\Omega^{p+q}(M,\mathbb{C}) \tag{B.2.5}$$

and

$$\Omega^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M,\mathbb{C}) \ . \tag{B.2.6}$$

A manifold with an AC structure is said to be complex if and only if for any $\omega \in \Omega^{p,q}(M,\mathbb{C})$

$$d\omega \in \Omega^{p+1,q}(M,\mathbb{C}) \oplus \Omega^{p,q+1}(M,\mathbb{C}) \ . \tag{B.2.7}$$

In this case we can decompose $d = \partial + \bar\partial$, where $\partial : \Omega^{p,q}(M,\mathbb{C}) \to \Omega^{p+1,q}(M,\mathbb{C})$ and $\bar\partial : \Omega^{p,q}(M,\mathbb{C}) \to \Omega^{p,q+1}(M,\mathbb{C})$. The latter operator $\bar\partial$ is known as the Dolbeault operator.

Exercise B.7. Using the definition just given, show that M is complex if and only if the almost complex structure has a vanishing Nienhuis tensor $\mathcal{N}^\mu_{\nu\rho}$ defined by

$$\mathcal{N}^\mu_{\nu\rho} = J^\alpha_\nu(\partial_\alpha J^\mu_\rho - \partial_\rho J^\mu_\alpha) - J^\alpha_\rho(\partial_\alpha J^\mu_\nu - \partial_\nu J^\mu_\alpha) \ . \tag{B.2.8}$$

Show that if $\dim M = 2$ then $\mathcal{N}(J) = 0$ for any almost complex structure.

An almost complex structure J with $\mathcal{N}(J) = 0$ is said to be an integrable almost complex structure , and a manifold admitting such a J is said to be complex.

Suppose M admits a holomorphic atlas. That is, M has a cover by open neighborhoods $U \simeq \mathbb{C}^n$, and holomorphic transition functions. In other words, given any two such neighborhoods U and V with $U \cap V \neq \emptyset$ and coordinates z^a, $a = 1, \ldots, n$ on U and w^a, $a = 1, \ldots, n$ on V, the $z^a(w^1, \ldots, w^n)$ are holomorphic and invertible functions on $U \cap V$. The differential forms in $\Omega^k(M, \mathbb{C})$ can then be decomposed in terms of the holomorphic and anti-holomorphic differentials dz^a and $d\overline{z}^{\overline{a}}$, and, indeed, M is complex by the definitions given above, since

$$J = idz^a \otimes \frac{\partial}{\partial z^a} - id\overline{z}^{\overline{a}} \otimes \frac{\partial}{\partial \overline{z}^{\overline{a}}} \tag{B.2.9}$$

is a complex structure.

Exercise B.8. Write $z^a = x^a + iy^a$ and $\overline{z}^{\overline{a}} = x^a - iy^a$ where x^a, y^a are real coordinates. We then have $dz^a = dx^a + idy^a$, and $\frac{\partial}{\partial z^a} = \frac{1}{2}\left(\frac{\partial}{\partial x^a} - i\frac{\partial}{\partial y^a}\right)$, and similarly for $d\overline{z}^{\overline{a}}$ and $\frac{\partial}{\partial \overline{z}^{\overline{a}}}$. Work out the form of J in the real coordinates.

A key result—the Newlander–Nirenberg theorem—asserts that every complex manifold M admits a holomorphic atlas. Indeed, given an almost complex structure J, its Nijenhuis tensor is the obstruction to finding a set of holomorphic coordinates in which J takes the canonical form of (B.2.9).

Compatibility of complex and symplectic structure

In general it is a difficult problem to show that a manifold with an AC structure admits either a complex or a symplectic structure. For instance, S^6 has an AC structure but no symplectic structure because $H^2(S^6, \mathbb{R})$ is empty; it is not known whether S^6 admits a complex structure. As another class of examples, it is known that every even-dimensional complex Lie group G, e.g. $G = S^3 \times S^3$, admits a complex structure but has no symplectic structure, again because $H^2(G, \mathbb{R}) = 0$.

An almost Hermitian geometry on M is a choice of an almost complex structure J and a Hermitian metric g. The latter is defined with respect to J by the requirement

$$g(J\cdot, J\cdot) = g(\cdot, \cdot) \ . \tag{B.2.10}$$

Such metrics are easy to construct: let h be any Riemannian metric on M; then

$$g(\cdot, \cdot) = h(\cdot, \cdot) + h(J\cdot, J\cdot) \tag{B.2.11}$$

is a Hermitian metric on M. Associated to the pair (J, g) is the Hermitian form $\omega \in \Omega^2(M, \mathbb{R})$ with components

$$\omega_{\mu\nu} = J_\mu^\lambda g_{\lambda\nu} \ . \tag{B.2.12}$$

Note that if our starting point is an AC structure J and an AS structure ω, then the two may not be compatible with a Hermitian metric: $\omega_{\mu\nu}J^{\nu}_{\lambda}$ need not be symmetric or positive-definite.

An almost Hermitian geometry with AC structure J and Hermitian form ω is said to be:

1. almost Kähler if $d\omega = 0$;
2. Hermitian if J is an integrable complex structure;
3. Kähler if J is an integrable complex structure and $d\omega = 0$.

The Hermitian and Kähler geometries are those of most relevance to (0,2) theories.

Differential forms on complex manifolds

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, and let z^a, $a = 1, \ldots, n$ denote local holomorphic coordinates. We denote their anti-holomorphic complex conjugates by $\overline{z^a} = \overline{z}^{\overline{a}}$, and we set

$$\partial_a = \frac{\partial}{\partial z^a} \ , \qquad\qquad \bar{\partial}_a = \frac{\partial}{\partial \overline{z}^a} \ . \tag{B.2.13}$$

A (p,q) differential form $\psi \in \Omega^{p,q}(M, \mathbb{C})$ is then given by (the wedges are understood)

$$\psi = \frac{1}{p!q!} \psi_{a_1 \cdots a_p \bar{b}_1 \cdots \bar{b}_q} dz^{a_1} \cdots dz^{a_p} d\overline{z}^{\bar{b}_1} \cdots d\overline{z}^{\bar{b}_q} \ . \tag{B.2.14}$$

The de Rham differential d splits as $d = \partial + \bar{\partial}$, where ∂ and $\bar{\partial}$ satisfy

$$\partial^2 = 0 \ , \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0 \ , \qquad \bar{\partial}^2 = 0 \ . \tag{B.2.15}$$

In local coordinates

$$\begin{aligned} \partial\psi &= \frac{1}{p!q!} \partial_{a_0} \psi_{a_1 \cdots a_p \bar{b}_1 \cdots \bar{b}_q} dz^{a_0} dz^{a_1} \cdots dz^{a_p} d\overline{z}^{\bar{b}_1} \cdots d\overline{z}^{\bar{b}_q} \ , \\ \bar{\partial}\psi &= \frac{(-1)^p}{p!q!} \bar{\partial}_{\bar{b}_0} \psi_{a_1 \cdots a_p \bar{b}_1 \cdots \bar{b}_q} dz^{a_0} dz^{a_1} \cdots dz^{a_p} d\overline{z}^{\bar{b}_0} \cdots d\overline{z}^{\bar{b}_1} d\overline{z}^{\bar{b}_q} \ . \end{aligned} \tag{B.2.16}$$

The Dolbeault operator $\bar{\partial}$ will play a particularly important role in the following.

We also define the complex conjugate (q,p) form $\overline{\psi}$ by

$$\begin{aligned} \overline{\psi} &= \frac{1}{q!p!} \overline{\psi_{a_1 \cdots a_p \bar{b}_1 \cdots \bar{b}_q}} d\overline{z}^{\bar{a}_1} \cdots d\overline{z}^{\bar{a}_p} dz^{b_1} \cdots dz^{b_q} \\ &= \frac{(-1)^{pq}}{q!p!} \overline{\psi_{a_1 \cdots a_p \bar{b}_1 \cdots \bar{b}_q}} dz^{b_1} \cdots dz^{b_q} d\overline{z}^{\bar{a}_1} \cdots d\overline{z}^{\bar{a}_p} \ . \end{aligned} \tag{B.2.17}$$

Note that complex conjugation defined this way commutes with differentials $\overline{\partial\psi} = \bar{\partial}\,\overline{\psi}$ and $\overline{\bar{\partial}\psi} = \partial\overline{\psi}$, as well as with the wedge product: $\overline{\alpha \wedge \beta} = \overline{\alpha} \wedge \overline{\beta}$. We are following here the convention of [77] that the "bar" can act on the indices of a tensor. This may at first be a bit confusing, but it is a very useful notation. To help clarify the point, the components of $\overline{\psi}$ may be written as

$$\overline{\psi}_{b_1\cdots b_q\overline{a}_1\cdots\overline{a}_p} = (-1)^{pq}(\overline{\psi})_{\overline{a}_1\cdots\overline{a}_p b_1\cdots b_q} = (-1)^{pq}\overline{\psi_{a_1\cdots a_p\overline{b}_1\cdots\overline{b}_q}} \ . \tag{B.2.18}$$

The object in the last equality is just the usual complex conjugate of the particular component of the tensor.

Hermitian metric and the Hodge star

The complexified tangent bundle on a complex manifold decomposes into a direct sum of a holomorphic and an anti-holomorphic component:

$$T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1} \ . \tag{B.2.19}$$

This is an instance of reduction of structure on the manifold: while the tangent bundle of a real manifold of dimension $2n$ has in general transition functions in $\mathrm{GL}(2n,\mathbb{R})$, a complex manifold has transition functions in $\mathrm{GL}(n,\mathbb{C})$.

For instance, sections of $T_M^{1,0}$ have the local form

$$V = V^a(z,\overline{z})\frac{\partial}{\partial z^a} \ , \tag{B.2.20}$$

and because M has holomorphic transition functions, this makes sense globally. We define the complex conjugate $\overline{V}$, as section of $T_M^{0,1}$ in the same way as we defined the complex conjugate differential forms above:

$$\overline{V} = \overline{V^a}\frac{\partial}{\partial \overline{z}^{\overline{a}}} \ . \tag{B.2.21}$$

A Hermitian metric on M may be thought of as a map $g : T_M^{1,0} \otimes T_M^{0,1} \to \mathbb{C}$

$$g = g_{a\overline{b}}dz^a \otimes d\overline{z}^{\overline{b}} \tag{B.2.22}$$

such that $g_{a\overline{b}} = \overline{g_{b\overline{a}}}$ is a positive-definite Hermitian matrix. This means that g gives a Hermitian inner product on $T_M^{1,0}$. For any two sections $V, W \in T_M^{1,0}$ we set

$$g(V,W) = V^a g_{a\overline{b}}\overline{W^b} \ , \tag{B.2.23}$$

and with our assumptions this satisfies $\overline{g(V,W)} = g(W,V)$ and $g(V,V) > 0$ for all $V \neq 0$. We will write $g^{\overline{b}a}$ for the inverse metric.

If we write $z^a = x^a + iy^a$ and label the x^a, y^a collectively by X^μ, we obtain the Riemannian metric on T_M and the line element

$$ds^2 = G_{\mu\nu}dX^\mu dX^\nu = g_{a\overline{b}}dz^a d\overline{z}^{\overline{b}} + \text{c.c.} \ . \tag{B.2.24}$$

With this normalization $g_{a\overline{b}} = \frac{1}{2}\delta_{a\overline{b}}$ leads to the standard Euclidean metric on $\mathbb{C}^n$.

The Hermitian form is

$$\omega = ig_{a\overline{b}}dz^a \wedge d\overline{z}^{\overline{b}} \ . \tag{B.2.25}$$

Exercise B.9. Compute the top power of the Hermitian form to obtain the Hermitian volume form

$$\frac{1}{n!}\omega^n = \det g \prod_{k=1}^{n} i dz^a \wedge d\bar{z}^a = 2^n (\det g) dx^1 \wedge dy^1 \cdots dx^n \wedge dy^n \ . \tag{B.2.26}$$

This gives a canonical orientation to M.

Check that on $\mathbb{C}^n$ with $g_{a\bar{b}} = \frac{1}{2}\delta_{a\bar{b}}$ we obtain the canonical form for g and ω:

$$\begin{aligned} ds^2 &= (dx^1)^2 + (dy^1)^2 + \cdots + (dx^n)^2 + (dy^n)^2 \ , \\ \omega &= dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n \ . \end{aligned} \tag{B.2.27}$$

With those preliminaries, we can now define a Hermitian inner product on (p,q) forms. Let $\alpha, \beta \in \Omega^{p,q}(M, \mathbb{C})$ and set

$$(\alpha, \beta) = \frac{1}{p!q!} \alpha_{a_1 \cdots a_p \bar{b}_1 \cdots \bar{b}_q} \overline{\beta_{c_1 \cdots c_p \bar{d}_1 \cdots \bar{d}_q}} g^{\bar{c}_1 a_1} \cdots g^{\bar{c}_p a_p} g^{\bar{b}_1 d_1} \cdots g^{\bar{b}_q d_q} \ . \tag{B.2.28}$$

Clearly $\overline{(\alpha, \beta)} = (\beta, \alpha)$, and $(\cdot, \cdot)$ is positive-definite point-wise on M.

The Hodge star is taken to be the unique map

$$* : \Omega^{p,q}(M, \mathbb{C}) \to \Omega^{n-q, n-p}(M, \mathbb{C}) \tag{B.2.29}$$

such that for any $\alpha, \beta \in \Omega^{p,q}(M, \mathbb{C})$

$$\alpha \wedge * \overline{\beta} = (\alpha, \beta) \frac{1}{n!} \omega^n \ . \tag{B.2.30}$$

Exercise B.10. Show that $*$ on $\Omega^{p,q}$ is given by

$$*\psi = \frac{i^n (-1)^{pn + n(n-1)/2} \det g}{p!q!(n-p)!(n-q)!} \psi^{\bar{b}_1 \cdots \bar{b}_p a_1 \cdots a_q} \epsilon_{a_1 \cdots a_n} \epsilon_{\bar{b}_1 \cdots \bar{b}_n} dz^{a_{p+1}} \cdots dz^{a_n} d\bar{z}^{\bar{b}_{p+1}} \cdots d\bar{z}^{\bar{b}_n} \ , \tag{B.2.31}$$

and verify

$$*^2 \Omega^{p,q}(M, \mathbb{C}) = (-1)^{p+q} \Omega^{p,q}(M, \mathbb{C}) \ , \tag{B.2.32}$$

as well as

$$*\overline{\psi} = \overline{*\psi} \ . \tag{B.2.33}$$

With this exercise in hand, we define the Hermitian inner product on (p,q) forms as

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) \frac{\omega^n}{n!} = \int_M \alpha \wedge * \overline{\beta} \ . \tag{B.2.34}$$

We could now proceed to define a refinement of de Rham cohomology that follows from the decomposition $d = \partial + \bar{\partial}$, but we will first pause to discuss a few notions of holomorphic vector bundles. These will allow us to build a much more general framework for this refined structure, known as Dolbeault cohomology.

Holomorphic vector bundles

A holomorphic vector bundle over a complex manifold M is simply a vector bundle that respects the underlying complex structure of M: the fibers are isomorphic to $\mathbb{C}^k$, where k is the rank of the bundle, and the transition functions are holomorphic functions of the coordinates. The tangent bundle $T^{1,0}_M$, its dual $\Omega^{1,0}_M$, or their tensor powers are all examples of holomorphic bundles.

A holomorphic vector bundle of rank 1 is known as a line bundle. Every complex manifold comes with a canonical line bundle given by the top exterior power of $\Omega^{1,0}_M$; it is frequenly denote by

$$K_M = \wedge^n \Omega^{1,0}_M \ . \tag{B.2.35}$$

For any holomorphic bundle $\mathcal{E}$ we can consider the bundle of $\mathcal{E}$–valued (p,q) differential forms, $\mathcal{E} \otimes \wedge^p (T^{1,0}_M)^* \otimes \wedge^q (T^{0,1}_M)^*$, and we denote the sections of such a bundle by $\Omega^{p,q}(M,\mathcal{E})$. Clearly $\Omega^{0,q}(M, \wedge^p(T^{1,0}_M)^* \otimes \mathcal{E}) = \Omega^{p,q}(M,\mathcal{E})$. The preceding discussion of p,q differential forms is then just a special case where we take $\mathcal{E}$ to be the trivial bundle over M. We will denote this by $\mathcal{O}$, and with a slight abuse of notation write $\Omega^{p,q}(M,\mathbb{C}) = \Omega^{p,q}(M,\mathcal{O})$.

On a general real vector bundle there is no canonical choice of connection and therefore no canonical differential operator. This is not the case for holomorphic bundles, which come with a canonical differential operator $\bar\partial$. Let η_A, $A = 1,\ldots,k$ be a holomorphic frame for the bundle $\mathcal{E}$: that is, on each patch the η_A form a basis for $\mathbb{C}^k$, and the η_A have holomorphic transition functions in $\mathrm{GL}(k,\mathbb{C})$ on overlaps. We then define $\bar\partial$ as follows:

$$\begin{aligned} &\bar\partial : \Omega^{0,0}(M,\mathcal{E}) \to \Omega^{0,1}(M,\mathcal{E}) \\ &\bar\partial : s = s^A \otimes \eta_A \mapsto \bar\partial s = d\bar z^{\bar a}(\bar\partial_{\bar a} s^A) \otimes \eta_A \ . \end{aligned} \tag{B.2.36}$$

We can extend the action to $\Omega^{0,q}(M,\mathcal{E})$ by enforcing the Leibniz rule. If

$$s = \frac{1}{k!} s^A_{\bar a_1 \cdots \bar a_k} d\bar z^{\bar a_1} \wedge \cdots d\bar z^{\bar a_k} \otimes \eta^A \in \Omega^{0,k}(M,\mathcal{E}) \ , \tag{B.2.37}$$

then

$$\bar\partial s = \frac{1}{k!} \bar\partial_{\bar a_0} s^A_{\bar a_1 \cdots \bar a_k} d\bar z^{\bar a_0} \wedge d\bar z^{\bar a_1} \wedge \cdots d\bar z^{\bar a_k} \otimes \eta_A \in \Omega^{0,k+1}(M,\mathcal{E}) \ . \tag{B.2.38}$$

For any holomorphic bundle $\mathcal{E}$ we have the corresponding anti-holomorphic bundle $\overline{\mathcal{E}}$: its sections are the complex conjugates of those of $\mathcal{E}$, and of course the transition functions for $\overline{\mathcal{E}}$ are anti-holomorphic. Therefore, it is the holomorphic differential ∂ that has a canonical action on sections of $\overline{\mathcal{E}}$.

Suppose that the holomorphic bundle $\mathcal{E}$ is equipped with a Hermitian metric. In other words, given $\mathcal{E}$ and its complex conjugate bundle $\overline{\mathcal{E}}$, we have h a section of $\mathcal{E}^* \otimes \overline{\mathcal{E}}^*$, such that for any two sections $s_1, s_2 \in \Gamma(\mathcal{E})$ we have

$$h(s_1,s_2) = \overline{h(s_2,s_1)} = h_{A\bar B} s_1^A \overline{s_2^B} \ , \tag{B.2.39}$$

and $h_{A\bar{B}}$ is a positive-definite Hermitian matrix. Using this, we can extend the definition of Hodge star and the inner product defined in (B.2.34) to sections in $\Omega^{p,q}(\mathcal{E})$:

$$\langle s_1, s_2 \rangle = \int_M s_1 \wedge *_{\mathcal{E}} \overline{s_2} = \int_M (s_1, s_2) \frac{\omega^n}{n!} \ , \tag{B.2.40}$$

where

$$(s_1, s_2) = \frac{1}{p!q!} (s_1)^A_{a_1 \cdots a_p \bar{b}_1 \cdots \bar{b}_q} \overline{(s_2)^B_{c_1 \cdots c_p \bar{d}_1 \cdots \bar{d}_q}} g^{\bar{c}_1 a_1} \cdots g^{\bar{c}_p a_p} g^{\bar{b}_1 d_1} \cdots g^{\bar{b}_q d_q} h_{A\bar{B}} \ . \tag{B.2.41}$$

Just as in our discussion of p,q forms, it is useful to think of the map $s \to *_{\mathcal{E}} \overline{s}$ as a composition of two isomorphisms:

$$\Omega^{p,q}(M, \mathcal{E}) \simeq \Omega^{q,p}(M, \overline{\mathcal{E}}) \simeq \Omega^{n-p,n-q}(M, \mathcal{E}^*) \ , \tag{B.2.42}$$

where the first isomorphism is obtained by complex conjugation, while the second one is the Hodge star $*_{\mathcal{E}}$:

$$*_{\mathcal{E}} : \Omega^{p,q}(M, \mathcal{E}) \to \Omega^{n-q,n-p}(M, \overline{\mathcal{E}}^*) \ . \tag{B.2.43}$$

The wedge operator in (B.2.40) should be understood to include the dual pairing between $\mathcal{E}$ and $\mathcal{E}^*$, i.e if $s_1 \in \Omega^{p_1,q_1}(M, \mathcal{E})$ and $s_2 \in \Omega^{p_2,q_2}(M, \mathcal{E}^*)$, then we set

$$s_1 \wedge s_2 = s_1^A \wedge s_{2A} \ . \tag{B.2.44}$$

We can now construct the formal adjoint of the Dolbeault operator. Demanding that for all $s \in \Omega^{p,q-1}(M, \mathcal{E})$ and $t \in \Omega^{p,q}(M, \mathcal{E})$

$$\langle \bar{\partial} s, t \rangle = \langle s, \bar{\partial}^\dagger t \rangle \ , \tag{B.2.45}$$

we find

$$\bar{\partial}^\dagger = - *_{\mathcal{E}} \partial *_{\mathcal{E}} \ . \tag{B.2.46}$$

Exercise B.11. Check the form of $\bar{\partial}^\dagger$.

In a completely analogous fashion to the Hodge-de Rham discussion, we define the $\bar{\partial}$–Laplace operator

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger \bar{\partial} \ , \tag{B.2.47}$$

which is formally self-adjoint on $\Omega^{p,q}(M, \mathcal{E})$ and commutes with $\bar{\partial}$ and $\bar{\partial}^\dagger$. Moreover, we have the relation

$$\Delta_{\bar{\partial}}(\overline{*_{\mathcal{E}} s}) = \overline{*_{\mathcal{E}} \Delta_{\bar{\partial}} s} \ . \tag{B.2.48}$$

On a compact manifold $\Delta_{\bar{\partial}}$ is elliptic, and it can again be established that it has a finite-dimensional kernel%REF. Letting $\Pi^{p,q}(\mathcal{E})$ denote the projector onto the space of zero modes of $\Delta_{\bar{\partial}}$ on $\Omega^{p,q}(M,\mathcal{E})$—we denote this subspace by $\mathcal{H}^{p,q}(\mathcal{E})$—we have another Green's operator $\Sigma_{\bar{\partial}}$ such that

$$\mathbb{1} = \Pi^{p,q}_{\bar{\partial}} + \Delta_{\bar{\partial}}\Sigma_{\bar{\partial}} \ , \tag{B.2.49}$$

and therefore the decomposition of $\Omega^{p,q}(M,\mathcal{E})$ into three orthogonal terms: the harmonic forms, the exact forms, and the co-exact forms:

$$s = \Pi^{p,q}_{\bar{\partial}} s + \bar{\partial}(\bar{\partial}^\dagger \Sigma_{\bar{\partial}} s) + \bar{\partial}^\dagger(\bar{\partial}\Sigma_{\bar{\partial}} s) \ . \tag{B.2.50}$$

All of this is entirely analogous to the Hodge decomposition on a real manifold discussed above, and there is also a cohomological parallel given by Dolbeault cohomology.

The Dolbeault cohomology groups are defined by

$$H^{p,q}_{\bar{\partial}}(M,\mathcal{E}) = \frac{\ker\{\bar{\partial} : \Omega^{p,q}(M,\mathcal{E}) \to \Omega^{p,q+1}(M,\mathcal{E})\}}{\operatorname{im}\{\bar{\partial} : \Omega^{p,q-1}(M,\mathcal{E}) \to \Omega^{p,q}(M,\mathcal{E})\}} \ , \tag{B.2.51}$$

and by the previous results we have the isomorphism $H^{p,q}_{\bar{\partial}}(M,\mathcal{E}) = \mathcal{H}^{p,q}(\mathcal{E})$. Moreover, the relation (B.2.48) implies the important Kodaira-Serre duality:

$$H^{p,q}_{\bar{\partial}}(M,\mathcal{E}) \simeq \overline{H^{n-q,n-p}_{\bar{\partial}}(M,\mathcal{E}^*)} \ . \tag{B.2.52}$$

There is an important difference between de Rham cohomology and Dolbeault cohomology. The former only depends on the differential structure on M and is a topological invariant of the manifold; the latter requires a choice of complex structure to define $\bar{\partial}$, and in general the cohomology groups can change as the complex structure is varied. Thus, unlike the Betti numbers b_k, the dimensions

$$h^{p,q}(M,\mathcal{E}) = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(M,\mathcal{E}) \tag{B.2.53}$$

vary with complex structure of M, but they do so in a way consistent with Kodaira-Serre:

$$h^{p,q}(M,\mathcal{E}) = h^{n-q,n-p}(M,\mathcal{E}^*) \ . \tag{B.2.54}$$

Čech cohomology

Since $\wedge^k\Omega^{1,0}_M \otimes \mathcal{E}$ is itself a holomorphic vector bundle, we may equivalently write

$$\Omega^{p,q}(M,\mathcal{E}) = \Omega^{0,q}(M, \wedge^p\Omega^{1,0}_M \otimes \mathcal{E}) \ . \tag{B.2.55}$$

This presentation is particularly useful because the right-hand-side can be identified with elements in Čech cochains that define the sheaf cohomology groups $H^q(M, \wedge^p\Omega^{1,0}_M \otimes \mathcal{E})$. These can be defined on a much more general class of spaces such as algebraic varieties where Dolbeault cohomology groups may not exist. However, when both exist we have the isomorphisms

$$H^q(M,\mathcal{E}) = H^{0,q}_{\bar\partial}(M,\mathcal{E}) \tag{B.2.56}$$

for any holomorphic bundle $\mathcal{E}$. More details on Čech cohomology may be found in any standard reference on algebraic or complex geometry, e.g. [72, 78]. This is a very useful relation. Among other things, it shows that the Dolbeault cohomology groups do not depend on the choice of Hermitian metric on M.

We will follow the literature and make the following slight abuse of notation:

$$h^q(M,\mathcal{E}) = \dim_{\mathbb{C}} H^q(M,\mathcal{E}) \ , \qquad h^{p,q}(M,\mathcal{E}) = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(M,\mathcal{E}) \ . \tag{B.2.57}$$

This usually does not cause confusion.

Exercise B.12. To gain some practice with the notation, recast the Kodaira-Serre isomorphism as

$$h^q(M,\mathcal{E}) = h^{n-q}(M,\mathcal{E}\otimes K_M) \ . \tag{B.2.58}$$

Hodge and Betti numbers and the $\partial\bar\partial$–lemma

We now return to the simple choice where we set $\mathcal{E} = \mathcal{O}$, the trivial bundle. In that case we speak of the Hodge numbers of M as $h^{p,q}(M) = h^{p,q}(M,\mathcal{O})$. In general there is no simple relation between these dimensions and the Betti numbers b_k. On an arbitrary compact complex manifold M the best one has is the Frölicher inequality

$$b_k \leq \sum_{p+q=k} h^{p,q}(M) \ . \tag{B.2.59}$$

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